COXETER GROUPS

Lent term

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M. Davis, geometry and topology of Coxeter groups

A. Thomas, Geometric and topological aspects of Coxeter groups and buildings

mostly following this one

Course outline

1. Geometric reflection groups
2. Defining abstract reflection groups
3. Combinatorics of Coxeter groups
4. The tits representation
5. Finite Coxeter groups
6. The basic construction
7. The Davis complex.

1] Geometric Reflection Groups

Coxeter groups are discrete groups generated by "reflections". In §2,3 we'll make this precise. In this section we'll see some examples.

Recall a Riemannian manifold is a smooth manifold M with a positive definite inner product on T_xM $\forall x \in M$.
This inner product allows us to define some notions:

- isometries: inner product preserving diffeomorphism
- metric: (distance)
- geodesics: (distance minimising curves)
- sectional curvature

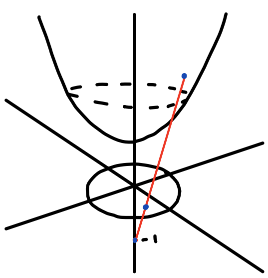
1.1. Notation

$\mathbb{S}^n :=$ n-dimensional sphere $\subset \mathbb{R}^{n+1}$ centred at origin with round metric
$\mathbb{E}^n :=$ n-dimensional Euclidean space $\cong (\mathbb{R}^n, \cdot)$
$\mathbb{H}^n :=$ n-dimensional real hyperbolic space
$\mathbb{X}^n :=$ any of these spaces $\mathbb{S}^n$, $\mathbb{E}^n$, or $\mathbb{H}^n$.

$\mathrm{Isom}(\mathbb{X}^n) :=$ isometry group of $\mathbb{X}^n$

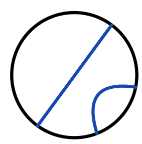
1.2. Remark: $\mathbb{S}^n$, $\mathbb{E}^n$ and $\mathbb{H}^n$ are all Riemannian manifolds with constant sectional curvature $1, 0, -1$ respectively.

Aside: Poincaré disc model for $\mathbb{H}^2$



$x^2 + y^2 + z^2 = 1$
$z > 0$

Poincaré disc model:
all points lie inside the unit disc.

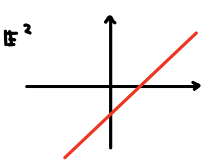


Geodesics on disc are semicircles or diameters.

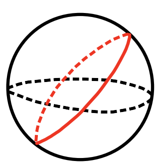
1.3. Definition: A hyperplane $\mathcal{H} \in \mathbb{X}^n$ is a totally geodesic, codimension 1 submanifold of $\mathbb{X}^n$

A hyperplane $\mathcal{H}$ separates $\mathbb{X}^n$ into two connected components, called half-spaces

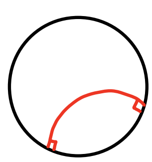
Ex: $\mathbb{E}^2$



$\mathbb{S}^2$:



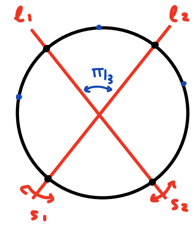
$\mathbb{H}^2$:



For each $\mathcal{H} \subset \mathbb{X}^n$, $\exists$ a reflection $\in \mathrm{Isom}(\mathbb{X}^n)$ which a) fixes $\mathcal{H}$ and b) exchanges the associated half-spaces.

1.4 Example : finite dihedral groups

S^1 with "hyperplanes" ℓ_1 and ℓ_2 meeting at angle $\frac{\pi}{m}$.



$S^1 \subset \mathbb{R}^2$

technically not hyperplanes because not totally geodesic.

Then $s_1 s_2$ is a rotation by $\frac{2\pi}{m}$, i.e. $\langle s_1 s_2 \rangle \cong C_m$

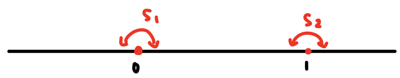
The group $W = \langle s_1, s_2 \rangle \cong D_{2m} = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^m = e \rangle$ identity

(group generated by reflections s_1 and s_2) (group generated by letters s_1, s_2)

is the dihedral group of order $2m$ (symmetries of the ngon).

1.5 Example : infinite dihedral group

$\mathbb{E}^1 = \mathbb{R}^1$



hyperplanes are points, e.g. 0 and 1

Let s_1 and s_2 be reflections about these points

i.e. $s_1(t) = -t$ and $s_2(t) = 2-t$ $\forall t \in \mathbb{R}$. Then the product $s_1 s_2$ is translation by 2 $(s_2 \circ s_1)$

means do s_1 then s_2.

i.e. $\langle s_1 s_2 \rangle \cong \mathbb{Z}$

and $W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = e \rangle = D_\infty$ is the infinite dihedral group.

Notation: if $\langle s_1 s_2 \rangle \cong \mathbb{Z}$, i.e. $s_1 s_2$ has infinite order, then we will write $(s_1 s_2)^\infty = e$.

Hence we can also write $W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^\infty = e \rangle$

1.7. Definition:

Let X be a topological space and $G \curvearrowright X$ by homeomorphisms. Let Gx be the orbit of $x \in X$.

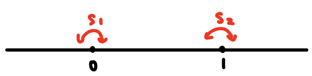
Then a **fundamental domain** for $G \curvearrowright X$ is $K \subset X$ s.t.

- K is closed and connected
- $Gx \cap K \neq \emptyset$ $\forall x \in X$
- $Gx \cap K = x$ if $x \in \operatorname{int}(K)$

K is known as a **strict fundamental domain** if $Gx \cap K = x$ $\forall x \in K$ (not just interior). That means that K contains exactly one point from each orbit.

1.8. Example

$\mathbb{E}^1$



$D_\infty \curvearrowright \mathbb{R}$

Then the closed interval $[0,1]$ is a strict fundamental domain, and so is any interval $[t, t+1]$ $\forall t \in \mathbb{R}$. The interval $[t, t+2]$ is a fundamental domain, but is not strict.

$s_1(t) = -t$
$s_2(t) = 2-t$

Rem: notation: $s_1 s_2 \cdot x = s_1(s_2(x))$

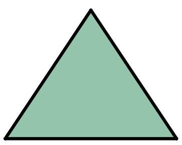
need to do a correction here.

Recall: simplex D^k is the convex hull of $(n+1)$ points, is called regular if any permutation of vertices can be realised by an isometry of $\mathbb{X}^n$

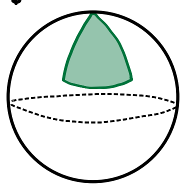
Dfn: a simplex $\sigma_k \subseteq \mathbb{X}^n$ for $k \leq n$ is the convex hull of $k+1$ basis vectors in $\mathbb{X}^n$. It is k-dimensional. A simplex is regular if all edges have the same length. Δ^k denotes the regular Euclidean k-simplex.

e.g. $n=2$, $k=2$ in

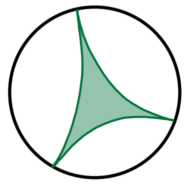
$\mathbb{E}^2$:



$\mathbb{S}^2$



$\mathbb{H}^2$



edges are same length

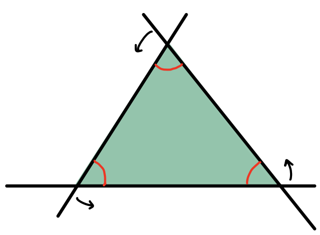
recall the half spaces are the connected components in the complement of the hyperplane.

1.9. Definition: a convex polytope $P \subseteq \mathbb{X}^n$ is a <u>convex</u>, compact intersection of a finite number of closed half spaces in $\mathbb{X}$ with nonempty interior

The link of a vertex v of P is $\text{link}_P(v) = P \cap$ unit $(n-1)$-sphere centred at v (sphere in $T_v\mathbb{X}^n$) This is a spherical $(n-1)$-dimensional polytope. P is called simple if $\forall\, v \in P$, $\text{link}_P(v)$ is a <u>regular</u> spherical simplex. Equivalently, P is called simple if each vertex is adjacent to exactly n edges.

not sure if just in $\mathbb{E}^n$

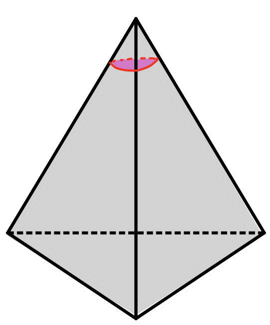
1.10 Example: $n=2$. A convex polytope in $\mathbb{X}^2$ is a convex polygon, and every convex polygon is simple.



arrows rep. choice of half spaces

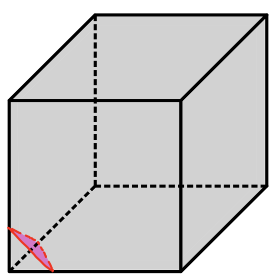
remember: polytope $\neq$ simplex. Below, we have 2-simplices in (a) and (b), but in (c) its not a simplex (2 dim. but <u>not</u> convex hull of 3 points).

$n=3$ $\quad P \subseteq \mathbb{E}^3$



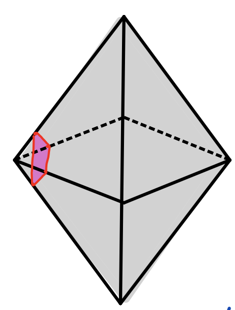
(a)

Simple



(b)

Simple



(c)

not simple. ($\text{link}_P(v)$ is not regular)

these are solid shapes.

1.11. Theorem (to prove later) $P \subseteq \mathbb{X}^n$ a simple convex polytope with $n \geq 2$. Let $\{F_i\}_{i \in I}$ be the set of codimension-1 faces of P. Then each F_i lies in an $\mathcal{H}_i \subseteq \mathbb{X}^n$.
Suppose $\forall i \neq j$ if $F_i \cap F_j \neq \emptyset$, then $\mathcal{H}_i$ and $\mathcal{H}_j$ intersect at an angle $\frac{\pi}{m_{ij}}$ where $m_{ij} \geq 2 \in \mathbb{Z}$.
Set $m_{ii} = 1$ and $m_{ij} = \infty$ when $F_i \cap F_j = \emptyset$, and s_i be reflection across $\mathcal{H}_i$ in $\mathrm{Isom}(\mathbb{X}^n)$
Let W be the group generated by $\{s_i\}_{i \in I}$. Then:

1) W has the following presentation: $W = \langle s_i : (s_i s_j)^{m_{ij}} = e \ \forall i,j \in I \rangle$

2) W is a discrete subgroup of $\mathrm{Isom}(\mathbb{X}^n)$

3) P is a strict fundamental domain for $W \curvearrowright \mathbb{X}^n$, and the action induces a tessellation (tiling) of $\mathbb{X}^n$ by copies of P.

1.12 Remark: Setting $m_{ii} = 1$ gives $(s_i s_i)^1 = e \Rightarrow s_i^2 = e \ \forall i \in I$.

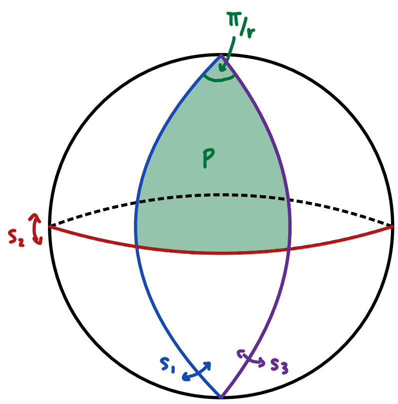
1.13. Definition: a group W is a geometric reflection group if it is D_{2m}, D_∞ or a group from Thm 1.11.
W is spherical if $\mathbb{X}^n = \mathbb{S}^n$, Euclidean if $\mathbb{X}^n = \mathbb{E}^n$, and Hyperbolic if $\mathbb{X}^n = \mathbb{H}^n$.

1.14 Remark: geometric reflection groups are our first examples of Coxeter groups. Coxeter classified all spherical and Euclidean groups in 1930s. Hyperbolic reflection groups are still not classified.

1.15 Examples: Triangle groups:
$\forall p,q,r \in \mathbb{Z}$ s.t $2 \leq p \leq q \leq r$, $\exists$ triangle $P \subsetneq \mathbb{X}^2$ with angles $\pi/p, \pi/q, \pi/r$. Then
$W = \langle s_1, s_2, s_3 : s_1^2 = s_2^2 = s_3^2 = e, (s_1 s_2)^p = (s_2 s_3)^q = (s_3 s_1)^r = e \rangle$

When $\mathbb{X}^2 = \mathbb{S}^2$, the angles of a triangle add up to $> 180°$ (π rad)
possible triples: $(2,2,r)$, $(2,3,3)$, $(2,3,4)$ and $(2,3,5)$

e.g. $(2,2,r)$



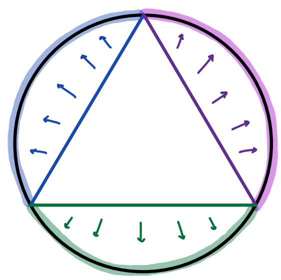
When $\mathbb{X}^2 = \mathbb{E}^2$, then $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} = \pi$, e.g. $p, q, r = 3$ Then

Example 1.16: Symmetric groups

For $n \geq 2$ and P a regular Euclidean simplex $D^n \subseteq \mathbb{E}^n$. Label the vertices with the set $\{1, \dots, n+1\}$, then $\mathrm{Isom}(D^n) \cong S_{n+1}$ the symmetric group on $n+1$ letters.

Embed $D^n \subseteq \mathbb{E}^n$ s.t vertices lie on $\mathbb{S}^{n-1}$, and then 'puff out' D^n to lie on $\mathbb{S}^{n-1}$, then we get a tessellation of $\mathbb{S}^{n-1}$ by the boundary ∂D^n

e.g. $n=2$:

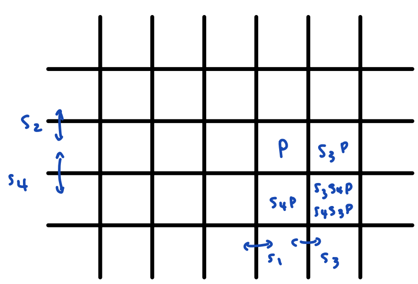


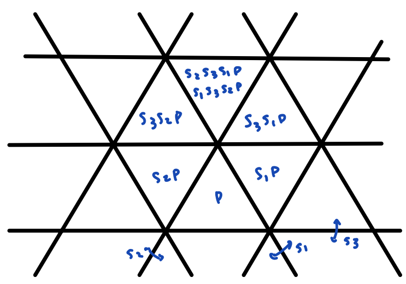
Take barycentric subdivision of ∂D^n, and letting P be maximal in this subdivision, then we get a presentation

$$W = S_{n+1} = \left\langle s_1, \dots, s_n \;\middle|\; \begin{array}{ll} s_i^2 = e & \\ (s_i s_j)^2 = e & |i-j| \geq 2 \\ (s_i s_{i+1})^3 = e & \text{for } 1 \leq i \leq n-1 \end{array} \right\rangle$$

Let $s_i = (i, i+1)$ gives S_{n+1} as permutation group.

Example 1.17: Tiling of $\mathbb{E}^n$ by n-cubes.





2. Defining Abstract reflection groups

2.1 Definition (Tits 1950s)

Let $S = \{s_i\}_{i \in I}$, I finite indexing set. A **Coxeter matrix** is a symmetric matrix $(S \times S)$, $M = (m_{ij})_{i,j \in I}$ such that the following hold

- $m_{ii} = 1 \quad \forall i \in I$
- $m_{ij} = m_{ji} \in \{2,3,4,\dots\} \cup \{\infty\} \quad \forall i \neq j$.

The **Coxeter group W** is the group

$$W = \langle S \mid (s_i s_j)^{m_{ij}} = e \quad \forall i,j \in I \rangle$$

and the pair **(W, S) is called a Coxeter system.**

2.2 Remark:

- Note $m_{ii} = 1 \Rightarrow s_i^2 = e \ \forall s_i \in S$. Also $(s_i s_j)^{m_{ij}}$ can be rewritten as $\underbrace{s_i s_j s_i s_j \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j s_i \cdots}_{m_{ij}}$
- Geometric reflection groups are Coxeter groups, but not all Coxeter groups are geometric reflection groups
- A Coxeter group W can correspond to multiple Coxeter systems.
  see isomorphism problem for Coxeter groups
- One can define (W, S) with $|S|$ infinite. We restrict ourselves to finite generating sets in this course.

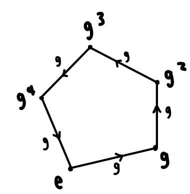
Next two corollaries are of Tit's representation (to come later)

2.3. Corollary : **if (W,S) is a Coxeter system, then the elements of S are**
• pairwise distinct and • involutions.
if (W,S) is a Coxeter system, then $\forall i \neq j$, $s_i s_j$ has order m_{ij} in W.

Let G be a group with generating set $S \not\ni e$.

2.4. Definition: The **Cayley graph of G** wrt. S, $\mathrm{Cay}_S(G)$ is the graph with vertex set $= G$. It has the directed edge set $\{(g, gs) : g \in G, s \in S, s^2 \neq e\}$ and undirected edge set $\{\{g, gs\} : g \in G, s \in S, s^2 = e\}$.
All edges are labelled by corresponding $s \in S$.

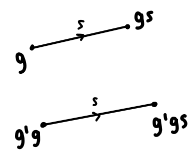
Ex: $G = C_5 = \langle g \rangle$. Then $\mathrm{Cay}_g(C_5)$:



In our examples, S is always a set of involutions (elements that square to identity), so all edges in $\mathrm{Cay}_S(G)$ will be undirected (really an edge in both directions)

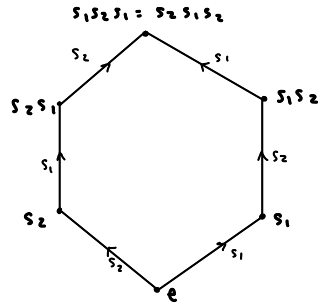
2.5 Remark. Since S generates G, **$\mathrm{Cay}_S(G)$ is connected** from Corollary 2.3. We also know for (W,S) a Coxeter system, $\mathrm{Cay}_S(W)$ is simple (no loops at vertex, and no double edges).

2.6. Lemma: G acts on $Cay_S(G)$ via multiplication on the left. This action preserves edge labels. Under this action, if $s^2 = e$, then gsg^{-1} is the unique group element which flips the edge $\{g, gs\}$.

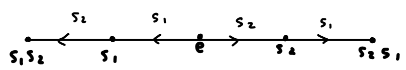


2.7 Example D_{2m}

e.g. D_6, $S = \{s_1, s_2\}$



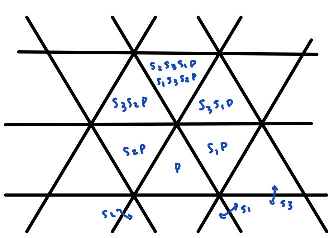
D_∞:

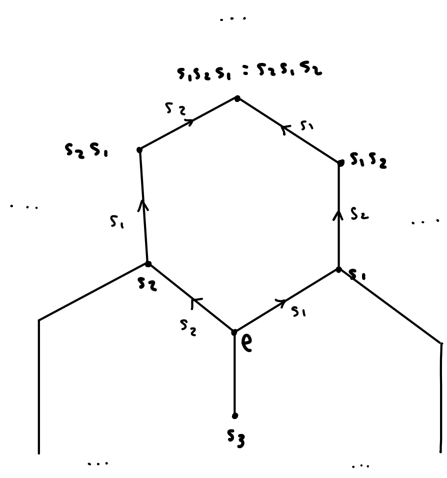


2.8 Remark: In case of geometric reflection groups, $Cay_S(W)$ is dual to the tessellation of $\mathbb{X}^n$ by the polytope P.

2.9. Examples

Triangle group $(3,3,3)$, $S = \{s_1, s_2, s_3\}$





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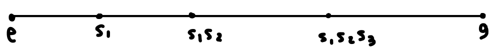
2.10 Definition: given a group G with generating set S of involutions, an **element** is a product of generators $s_1, \dots, s_n$, $s_i \in S$, and a **word** is a finite sequence of generators $(s_1, \dots, s_n)$ (care particularly about the order) $s_i \in S$. The **word length** of $g \in G$ wrt S is $\ell_S(g) = \min\{ n \in \mathbb{N} \mid g = s_1 \dots s_n \ \ s_i \in S \}$, and we set $\ell_S(e) = 0$. If $\ell_S(g) = n \geq 1$ and $g = s_1 \dots s_n$, then the sequence $(s_1, \dots, s_n)$ is called a **reduced word** for g. generators have length 1 by construction. Note it depends on choice of generating set.

e.g. in D_6, $s_1 s_2 s_1$ is an element, $s_1 s_2 s_1 = s_2 s_1 s_2$. (s_1, s_2, s_1) and (s_2, s_1, s_2) are reduced words for this element.

2.11 Definition: The **word metric** on G is given by $d_S(g,h) := \ell_S(g^{-1}h)$ for $g, h \in G$. This extends to a **path metric** on $Cay_S(G,S)$. Each edge is given length 1. The distance between two vertices is the shortest path between them.

2.12 Example: $d_S(e,g) = \ell_S(e^{-1}g) = \ell_S(g) = k$, and if $(s_1, \dots, s_k)$ is a reduced word for g, then we get a path of length k from e to g in $Cay_S(G)$



and this path has minimal length

2.13 Definition: A **pre-reflection system** for a group G is a pair (X, R) such that:

- X is a connected, simple graph
- $G \curvearrowright X$ by graph automorphisms
- R is a subset of G and
  - a) **every $r \in R$ is an involution** (i.e. $r^2 = e$
  - b) R is closed under conjugation : $\forall g \in G$, $r \in R$, $grg^{-1} \in R$
  - c) R generates G
  - d) $\forall \{v, w\} \in E(X)$ $\exists!$ $r \in R$ which flips $\{v, w\}$ (ie. interchanges v and w)
  - e) each $r \in R$ flips at least one edge.

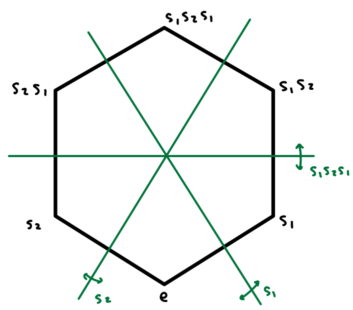
For $r \in R$, let $H_r = \{$ midpoints of edges flipped by $r\}$

2.14 Example

$W := D_6 = \{ s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^3 = e\}$ $\quad S = \{s_1, s_2\}$

Take $X = Cay_S(W)$

$R = \{ s_1, s_2, s_1 s_2 s_1 = s_2 s_1 s_2\}$

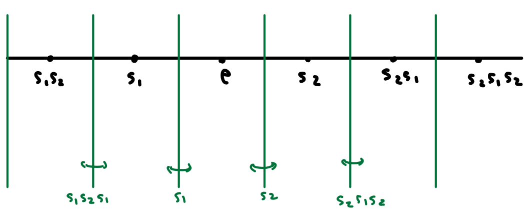


Then (X, R) is a prereflection system for $W = D_6$.

$W = D_\infty = \{ s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^\infty = e\}$ $\quad S = \{s_1, s_2\}$

$X = Cay_S(W)$

$R = \{ w s_i w^{-1} \mid s_i \in S, w \in W\}$



(X, R) is a prereflection system for D_∞

2.15. Lemma. If (X,R) is a prereflection system for G, then G acts transitively on $V(X)$.

proof: X is connected so $\exists$ path between any two vertices v and w: $(v = v_0, v_1, \ldots, v_k = w)$. Let r_i be the unique involution which flips $\{v_i, v_{i+1}\}$. Then $\underbrace{r_{k-1} r_{k-2} \cdots r_0}_{\in G} v = w$ $r_{k-1} \cdots r_0$ sends v to w $\square$

2.16. Lemma: let (W,S) be a Coxeter system and $R = \{ wsw^{-1} \mid s \in S, w \in W\}$. Then $(Cay_S(W), R)$ is a prereflection system for W.

proof: From Rem 2.5, $Cay_S(W)$ is always a connected simple graph. Also $(wsw^{-1})^2 = wsw^{-1}wsw^{-1} = ws^2w^{-1} = ww^{-1} = e$ $\forall\, wsw^{-1} \in R$, so they're involutions. Moreover, wsw^{-1} is the unique reflection which flips the edge $\{w, ws\} \in Cay_S(W)$

$$w \overset{s}{\text{———}} ws$$
$$= wsw^{-1}w \overset{s}{\text{———}} wsw^{-1}ws$$
$$= ws \overset{s}{\text{———}} w$$

2.17 Definition: Let (X,R) be a prereflection system for G. Then (X,R) is a reflection system if in addition it satisfies
 f) for each $r \in R$, $X \setminus H_r$ has exactly two components.

3. Combinatorics of Coxeter Groups

In this section, we will prove

3.1 Theorem: Let W be a group generated by a set S of distinct involutions. Then the following are equivalent:
(1) (W,S) is a Coxeter system
(2) Let $X = Cay_S(W)$, $R = \{ wsw^{-1} \mid s \in S, w \in W\}$. Then (X,R) is a reflection system
(3) (W,S) satisfies the 'deletion' condition
(4) (W,S) satisfies the 'exchange' condition

3.2. Definition: The pair (W,S) is said to satisfy the deletion condition if the following holds
 (D) if $w = (s_1, \ldots, s_k)$ is a word in S with $\ell_S(s_1 \ldots s_k) < k$, then $\exists$ indices $i<j$ such that $s_1 \cdots s_k = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$, where $\hat{s}_i$ means delete the letter s_i. (can delete 2)

3.3. Definition: The pair (W,S) is said to satisfy the exchange condition if the following holds
 (E) If $(s_1, \ldots, s_k)$ is a reduced word, then for any $s \in S$, either $\ell_S(ss_1 \ldots s_k) = k+1$, or $w = s_1 \cdots s_k = ss_1 \cdots \hat{s}_i \cdots s_k$ for $i \in \{1, \ldots, k\}$
 i.e. makes another reduced word.
 if this doesnt happen, then $\ell_S(ss_1 \ldots s_k) = k-1$.

Proof of Thm 3.1:

(3) $\Rightarrow$ (4) (deletion $\Rightarrow$ exchange). Suppose $(s_1, \ldots, s_k)$ for $w = s_1 \cdots s_k$ and $s_0 \in S$. Then

$$\begin{aligned} \ell_S(s_0 s_1 \cdots s_k) &= \ell_S(s_0 w) \\ &\le \ell_S(s_0) + \ell_S(w) \\ &= k+1. \end{aligned}$$

If $= k+1$ then we're done. So suppose $\ell_S(s_0 \cdots s_k) < k+1$ for contradiction's sake. By (D), $\exists$ indices $0 \le i,j \le k$ such that $s_0 w = s_0 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$. Since our original word $(s_1, \ldots, s_k)$ is reduced, we must have $i = 0$, otherwise multiplying on the left by s_0 gives $\lightning$. So $s_0 w = \hat{s}_0 \cdots \hat{s}_j \cdots s_k = s_1 \cdots \hat{s}_j \cdots s_k$. Multiplying on the left by s_0 gives $w = s_0 s_1 \cdots s_k$. Hence (W,S) satisfies the exchange condition.

We now prove some lemmas needed for (1) $\Rightarrow$ (2) $\Rightarrow$ (3). First, a discussion.

Discussion: let W be generated by S as in Thm 3.1. Then $\exists$ a bijection

$$\{\text{words in } S\} \longleftrightarrow \left\{\begin{array}{l}\text{paths in Cayley graph of } W \text{ gen by } S \\ \text{that start at the identity}\end{array}\right\}$$

Ex 2.12: $e \overset{s_1}{—} s_1 \overset{s_2}{—} s_1s_2 — \cdots — s_1 \cdots s_k$

Let $R = \{wsw^{-1} : w \in W, s \in S\}$. From lemma 2.6, $\exists!\ r \in R$ which flips each edge

$$s_1 \cdots s_{j-1} —— s_1 \cdots s_j$$

given by $r_j = s_1 \cdots s_j \, s_{j-1} \cdots s_1$. For example, $r_1 = s_1$, $r_2 = s_1 s_2 s_1$, $r_3 = s_1 s_2 s_3 s_2 s_1$

So we get a reflection sequence $(r_1, \dots, r_k)$ for a word $(s_1, \dots, s_k)$

If $H_r = \{\text{midpoint } \{v,w\} : r \text{ flips } \{v,w\}\}$, then we say $(s_1, \dots, s_k)$ crosses H_r if the associated path in $\mathrm{Cay}(W)$ contains an edge flipped by r. So $(s_1, \dots, s_k)$ crosses $H_{r_1}, \dots, H_{r_k}$

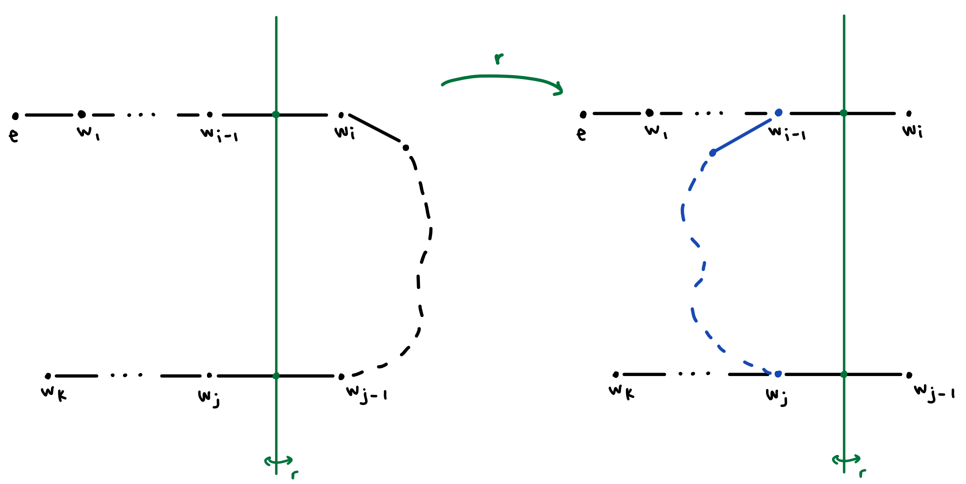
> $(s_1, \dots, s_k)$ has associated path $e — s_1 — s_1s_2 \overset{E_3}{—} s_1s_2s_3 — \cdots — s_1 \cdots s_k$. The edge E_3 is $s_1s_2 — s_1s_2s_3$, and is flipped by left-acting by $r_3 = s_1s_2s_3s_2s_1$:
>
> $$(s_1s_2s_3s_2s_1)(s_1s_2) — (s_1s_2s_3s_2s_1)(s_1s_2s_3) = s_1s_2s_3 — s_1s_2$$
>
> This element r is unique. Similarly, every E_i is uniquely flipped by r_i.

3.4. Lemma: let W, S, R be as above, and $(s_1, \dots, s_k)$ a word in S with associated reflection sequence $(r_1, \dots, r_k)$ such that $(r_1, \dots, r_k)$ s.t. $r_i = r_j$ for some $1 \le i \le j \le k$. Then in W,

$$s_1 \cdots s_k = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$$

proof: Let $r = r_i = r_j$ and $w_p = s_1 \cdots s_p$. Then in $\mathrm{Cay}_S(W)$ we have.

Applying the reflection r to the path $w_i \cdots w_{j-1}$ to get a path from w_{i-1} to w_j



idea is to build a path that gets you to the right element. If it crosses over twice, then we can just reflect it and ignore s_i and s_j.

The action of $W \curvearrowright \mathrm{Cay}_S(W)$ preserves edge labels so we get a new path to w_k:

$$(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_{j-1}, s_{j+1}, \dots, s_k) \qquad \text{(get rid of } s_i \text{ and } s_j \text{ edge)}$$

as required. $\square$

3.5. Lemma: With W, S, R as above, then for each $r \in R$, $Cay_S(W) \setminus H_r$ has at most two connected components.

proof: $r = wsw^{-1}$ for some $w \in W$, $s \in S$.

Claim: $w \cdot H_s = H_{wsw^{-1}} = H_r$

Sketch: if s flips edge $g \overset{s'}{\text{———}} gs'$, then wsw^{-1} flips $wg \overset{s'}{\text{———}} wgs' = w \cdot \left(g \overset{s'}{\text{———}} gs' \right)$

so $w \cdot H_s \subseteq H_{wsw^{-1}}$

other side comes from the fact that the action of W on edges is transitive

Then WLOG we can prove the lemma for H_s (because $W \curvearrowright Cay_S(W)$ by isometries). First, we show for all $v \in V(Cay_S(W)) = W$, then either v or sv is in the same component of $Cay_S(W) \setminus H_s$ as e.

Let $(s_1, \dots, s_k)$ be a reduced word for v $\rightsquigarrow$ we get a path in $Cay_S(W)$ from e to v with associated reflection sequence $(r_1, \dots, r_k)$. If $s \neq r_i$ for any i, then $\Rightarrow$ e and v are in the same component of $Cay_S(W) \setminus H_s$

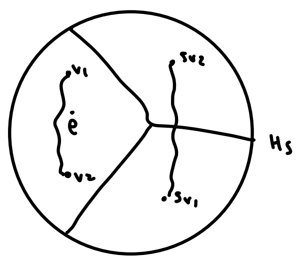
> To see this, notice that if $s \neq r_i$ for any i, then s does not reflect an edge of the path associated to v (by uniqueness of the r_i, each r_i is the only element to flip edge i). So the path associated to v cannot cross H_s. I.e. you start at e, and end at v, and this path is contained in one component of $Cay_S(W) \setminus H_s$

Now suppose $s = r_i$ for some i. Then by Lemma 3.4, since $(s_1, \dots, s_k)$ is reduced, $r_j \neq s$ $\forall i \neq j$. Then the word $(s, s_1, \dots, s_k)$ for sv has a reflection sequence $(s, r_1', \dots, r_k')$ where $r_j' = sr_js$.

> Remember $r_1 = s_1$, $r_2 = s_1 s_2 s_1$, $r_3 = s_1 s_2 s_3 s_2 s_1$, we're just adding a new first term.

Then $r_i' = s \overset{?}{=} s$, and $r_j' \neq s$ for $i \neq j \neq 1$. So we have $(s, r_1', \dots, r_k')$ has exactly two instances of s. Hence we can apply lemma 3.4 and delete to get a word for sv which corresponds to a path from e to sv not crossing H_s. Hence e and sv are in the same component.

Claim: we are done. Why? Suppose that $Cay_S(W) \setminus H_s$ has > 2 components. Then there exist $v_1 \neq v_2$ in the same component as e s.t sv_1 and sv_2 are in 2 other components (components are nonempty so can construct such v_1 and v_2) Now every path from sv_1 to sv_2 has to cross H_s $\Rightarrow$ every path from v_1 to v_2 crosses $sH_s = H_s$ $\lightning$.



$\square$

Let (W, S) be a Coxeter system. For any word $(s_1, \dots, s_k) = \underline{s}$, let $n(r, \underline{s})$ be the number of times the corresponding path crosses H_r in $Cay_S(W)$.

3.6. Lemma: (i) for any word $\underline{s} = (s_1, \dots, s_k)$ with $w = s_1 \cdots s_k$, then for any $r \in R$, $(-1)^{n(r, \underline{s})} \in \{\pm 1\}$ depends* only on $w \in W$ (not the word representation but the actual element).

(ii) $\exists$ a group homomorphism $W \longrightarrow Sym(R \times \{\pm 1\})$; $w \longmapsto \phi_w$ s.t $\phi_w(r, \varepsilon) = (wrw^{-1}, (-1)^{n(r, \underline{s})}\varepsilon)$, where $\underline{s}$ is any word representing w.

* So we can say that if $n(r, \underline{s})$ = odd for one word representing $\underline{s}$ for an element t, then $n(r, \underline{q})$ is also odd for any other word $\underline{q}$ for t.

proof: First we'll define ϕ for words, then show it extends to a group homomorphism and this will show (i) as well as (ii).

For $s \in S$, let $\phi_s \in \mathrm{Sym}(R \times \{\pm 1\})$ be given by $\phi_s(r, \varepsilon) = (srs, (-1)^{\delta_{rs}} \varepsilon)$, where $\delta_{rs} = \begin{cases} 1 & r = s \\ 0 & r \neq s \end{cases}$

We can check that ϕ_s is a bijection, since $\phi_s \circ \phi_s = \mathrm{id}_{R \times \{\pm 1\}}$ (any involution is a bijection)

We can extend this definition to words: If $\underline{s} = (s_1, \dots, s_k)$ is a word, then we define $\phi_{\underline{s}} \in \mathrm{Sym}(R \times \{\pm 1\})$ to be

$$\phi_{\underline{s}} = \phi_{s_k} \circ \dots \circ \phi_{s_1}$$

and can show inductively that $\phi_{\underline{s}}(r, \varepsilon) = (s_k \dots s_1 r s_1 \dots s_k, (-1)^{n(r, \underline{s})} \varepsilon)$

Let's check that this definition induces a homomorphism $W \to \mathrm{Sym}(R \times \{\pm 1\})$. We want to show that if $\underline{s}$ is a word for $(s_i s_j)^{m_{ij}}$, m_{ij} finite, then $\phi_{\underline{s}}$ is trivial (i.e. respects relations of W)

- $i = j$. Then $\underline{s} = (s, s)$, $\phi_{\underline{s}} = \phi_s \circ \phi_s = \mathrm{id}$ $\forall s \in S$ ✓
- $i \neq j$. Then $\underline{s} = (\underbrace{s_i, s_j, s_i, s_j \dots s_i, s_j}_{2m_{ij} \text{ letters}})$

of form wrw^{-1}

Then $\phi_{\underline{s}}(r, \varepsilon) = \underbrace{s_j s_i \dots s_j s_i}_{2m_{ij} \text{ letters}} r \underbrace{s_i s_j \dots s_i s_j}_{2m_{ij} \text{ letters}} = (s_j s_i)^{m_{ij}} r (s_i s_j)^{m_{ij}} = e r e = r.$

We also have to show that $n(r, S)$ is even, $\forall r \in R$. We'll deal with two cases. Notice that $\langle s_i, s_j \rangle \leq W$ is a subgroup isomorphic to D_{2m} where $m \mid m_{ij}$. (Not sure why not just m_{ij})

If $r \notin \langle s_i, s_j \rangle$, then $\underline{s} = (s_i s_j)^{m_{ij}}$ has a path that does not cross H_r. If it did, then r would flip some edge. But remember how we defined r:

if we have an edge say $s_i s_j s_i$ ⟶ $s_i s_j s_i s_j$

Then the unique r flipping it is $s_i s_j s_i s_j s_i s_j s_i$ because $(s_i s_j s_i s_j s_i s_j s_i)(s_i s_j s_i)$
$= s_i s_j s_i s_j s_i s_j s_i s_i s_j s_i$
$= s_i s_j s_i s_j$

But of course, this is made of s_i and s_j. But $r \notin \langle s_i, s_j \rangle$. Therefore $n(r, \underline{s}) = 0$.

In the case that $r \in \langle s_i, s_j \rangle$, then we know that r is some $s_i s_j \dots s_i s_j s_i \dots s_j s_i$

I'm not actually sure!

If $r \in \langle s_i, s_j \rangle$ then $n(r, \underline{s}) = \frac{2m_{ij}}{m}$ which is even. $\square$

If $\underline{s}$ is a word for $(s_i s_j)^{m_{ij}}$, we know that if say $r = (s_i s_j)^m s_i$ wlog, then $(s_i s_j)^m$ must appear in the word: remember r is the unique edge that flips

$s_i s_j s_i$ ⟶ $s_i s_j s_i s_j$

$s_i s_j s_i s_j s_i s_j s_i$

But then actually if $(s_i s_j)^m$ appears in the word, then since $\underline{s}$ is a word for $(s_i s_j)^{m_{ij}}$, we must have that $m = n m_{ij}$.

We know that if $r \in \langle s_i, s_j \rangle$, then it has to be of the form $s_i s_j s_i \cdots s_j s_i s_j \cdots s_i = (s_i s_j)^m s_i$ (wlog, could start with j). Anyway, then

$r = w s_i w^{-1}$ for some $w \in \langle s_i, s_j \rangle$. And in particular,

$\hookrightarrow$! edge flipping w — $w s_i$

what happens if w is a subword starting $\underline{s}$?

Idea: you have your Cayley graph, and you have a closed path in the cayley graph corresponding to the word $\underline{s}$ for $(s_i s_j)^{m_{ij}}$ which is just e.

possibly repetition

A word $\underline{s}$ for $(s_i s_j)^{m_{ij}} = e$ corresponds to any closed path in $\mathrm{Cay}_S(W)$, starting and ending at e. We know that the path is arbitrary and so can involve some s_k's with $k \neq i, j$. We know that if you have some $r \in \langle s_i, s_j \rangle$ flipping an edge of this path, then since R is a reflection system, r flips this edge and another edge e' of the cayley graph

Proof of Theorem 3.1 : (1) ⇒ (2) : A Coxeter system (W,S) gives a reflection system where $X = \mathrm{Cay}_S(W)$ and $R = \{ wsw^{-1} : w \in W,\ s \in S \}$.

By lemma 2.16, we already know that (X,R) is a prereflection system for W. So we only need to show that condition (f) holds: f) for each $r \in R$, $X \setminus H_r$ has exactly two components.

By lemma 3.5 we know that $X \setminus H_r$ has at most two components for each $r \in R$. So the claim follows if we can show that H_r separates X (then it must have more than one component). WLOG, similarly to before we only need to show this for H_s : we saw that if $r = wsw^{-1}$, then $H_r = w \cdot (H_s)$, and since $W \curvearrowright X$ via isometries, then if H_s separates the space, then so does H_r.

So let $r = s$. By lemma 3.6, since $n(r,s) = n(s,s) = 1$ (have $e \overset{s}{\text{———}} s$), for any path from e to s in X crosses H_s an odd number of times. This is because $(-1)^{n(r,w)} = (-1)^{n(s,w)}$ is independent of any choice of word w for s, and so we can just pick $(-1)^{n(r,w)} = (-1)^{n(r,s)} = (-1)^{n(s,s)} = (-1)^1 = -1$, so an odd number of times. In particular, it must cross H_s at least once. It follows that e and s lie then in separate components of $X \setminus H_r$

> *Any path from e to s crosses H_r an odd number of times, so every path $e \to s$ is split by H_r.* $\square$

Proof of Thm 3.1 : (2) ⇒ (3) : Says that if (X,R) is a reflection system, then it satisfies the deletion condition.

Recall the deletion condition says that if a word is not reduced, you can delete 2 generators from its word and still get the same element.

RECALL: **3.4. Lemma:** let W, S, R be as above, and $(s_1, \dots, s_k)$ a word in S with associated reflection sequence $(r_1, \dots, r_k)$ such that $(r_1, \dots, r_k)$ s.t $r_i = r_j$ for some $1 \leq i \leq j \leq k$. Then in W,

$$s_1 \dots s_k = s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_k$$

So if we can show that $\underline{s}$ is a reduced word $\Leftrightarrow$ r_i and r_j are pairwise distinct, then the claim will follow.

> *So if $\underline{s}$ is not reduced, then $\exists\ i \neq j$ s.t $r_i = r_j$ and then lemma 3.4 $\Rightarrow s_1 \dots s_k = s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_k$.*

(⇒) follows from Lemma 3.4. We're interested in the converse:

(⇐) Let $w = s_1 \dots s_k$ and $R(e,w) := \{ r \in R :$ e and w are in distinct components of $X \setminus H_r \}$. (components whose hyperplanes separate e and w)

Then for $r \in R(e,w)$, any path from e to w must cross H_r at least once. Hence r must be in the reflection sequence for w, i.e. $r = r_i$ for $1 \leq i \leq k$.

Any path, including the reduced word path from e to w must cross H_r for all $r \in R(e,w)$. And so every $r \in R(e,w)$ must be in the reflection sequence for the reduced word for w. Hence

$$\ell(w) \geq |R(e,w)|.$$

But we assumed that (X,R) is a reflection system, so $X \setminus H_r$ has two components for every r_i in the reflection sequence for $w = s_1 \dots s_k$. , of which there are k distinct reflections by assumption. (of course remember that r_i live in R by construction, they are of the form $w s_i w^{-1}$ for $w = s_1 \dots s_{i-1}$)

Now, the path $w = s_1 \dots s_k$ crosses H_{r_i} for $i = 1, \dots, k$, and particular (I think) it does so only once, because the r_i are pairwise distinct. Yup I think that's correct. And therefore e and w must lie in separate components of $X \setminus H_{r_i}$ $\forall i = 1, \dots, k$. Hence $|R(e,w)| \geq k$.

Therefore $k \geq \ell(w) \geq |R(e,w)| \geq k$, $\Rightarrow$ $\ell(w) = k$, so that $(s_1, \dots, s_k)$ is a reduced word for w.

$\square$

All that is left now is to prove that (4) $\Rightarrow$ (1), i.e. that W satisfying the exchange condition $\Rightarrow$ (W,S) is a Coxeter system. To do so, we state and prove Tits' solution to the word problem. This will take a little while, so hold on to your horses

3.7 Definition: Let W be generated by a set of distinct involutions S and $s \neq t \in S$ such that the order of st, m_{st}, is finite. A ***braid move*** on a word in S swaps a subword $(s, t, s, t, \dots)$ of length m_{st} with a subword $(t, s, t, s \dots)$ of length m_{st}.

3.8 Remark: • Since $(st)^{m_{st}} = e$ and $s^2 = t^2 = e$, carrying out a braid move does not change the group element which a word represents. $(st)^{m_{st}} = e \Rightarrow stst\dots st = t^{-1}s^{-1}t^{-1}s^{-1}\dots t^{-1}s^{-1} = tsts\dots ts$

• 'Braid move' comes from relations in the braid group, which are alternating relations of length 2 and 3.

3.9. Example: question: do they have to be right next to each other in the word? I guess not

In $D_6 = \langle s_1, s_2 : s_1^2 = s_2^2 = (s_1 s_2)^3 = e \rangle$, braid moves are given by swapping $(s_1, s_2, s_1) \leftrightarrow (s_2, s_1, s_2)$

In $D_\infty = \langle s_1, s_2 : s_1^2 = s_2^2 = e \rangle$, there are no braid moves.

Suppose $\langle S \mid R \rangle$ is a presentation for a group G. The ***word problem*** for $\langle S \mid R \rangle$ is the following:

Given $\underline{s}$ a word in $S \cup S^{-1}$, is there an algorithm for determining if the element it represents in G is the identity?

3.10 Theorem (Tits)

Suppose W is a group generated by a set S of distinct involutions, and (W,S) satisfies (E). Then

(1) a word $(s_1, \dots, s_k)$ is reduced $\Leftrightarrow$ it cannot be shortened by a sequence of
 (i) deleting a subword (s,s) $s \in S$, or
 (ii) a braid move.

(2) Two reduced words in S represent the same element $w \in W$ $\Leftrightarrow$ they are related by a finite sequence of braid moves.

(s_1, s_2, s_4, s_3) : (s_1, s_2) is a subword, but (s_1, s_3) not.

proof : *proof of 2 :*

$\Rightarrow$ Suppose we have reduced words $\underline{s} = (s_1, \dots, s_k)$ and $\underline{t} = (t_1, \dots, t_k)$, both representing $w \in W$. We'll do a proof by induction on $k = \ell(w)$.

Base: if $k=1$, then $\underline{s} = (s) = \underline{t}$ for some generator $s \in S$, and we're done.

Ind. hyp: assume true for elements w' such that $\ell(w) \le k-1$.

If $s_1 = t_1 = s$, then sw is represented by $(s_2, \dots, s_k)$ and $(t_2, \dots, t_k)$. Note that $\underline{s}$ and $\underline{t}$ are actually reduced: If sw is not reduced, then $\exists$ a repn $(q_1, \dots, q_j)$ with $j < k-1$, and then $(s, q_1, \dots, q_j)$ will be a word for $ssw = w$ with length $j+1 < k-1+1 < k$, a contradiction since $(s_1, \dots, s_k)$ is reduced. So $(s_2, \dots, s_k)$ is a reduced word for sw and so is $(t_2, \dots, t_k)$. By inductive hyp, we can transform one into the other by braid moves and hence we are done.

But what if $s_1 \ne t_1$? In that case, let $s_1 = s$ and $t_1 = t$.
Claim: m_{st} is finite, and $\exists$ a word $\underline{u} = (u_1, \dots, u_k)$ representing w starting with $(s,t,s,t \dots)$ of length m_{st}.

notice length k, i.e. u is reduced

Given the claim, let $\underline{u}'$ be such that $\underline{u} \rightsquigarrow \underline{u}'$ via braid move on the initial subword. Then we have:

$$\underline{s} \xrightarrow[\text{braid move}]{} \underline{u} \xrightarrow[\text{braid move}]{} \underline{u}' \xrightarrow[\text{braid move}]{} \underline{t}$$

where the first and last arrows are from the case where words start with the same letter.

proof of claim: Since w can start with the letter s or t, $\ell(tw) < \ell(w)$ (using reduced word stuff like before) and by (E) this means that $s_1 s_2 \dots s_k = t s_1 \dots \hat{s_i} \dots s_k$ for some $1 \le i \le k$ (remember exchange condition says that tacking on a generator to the front either increases the length of the word by 1, or we can exchange the generator for one in the word)

Okay, so $s_1 \dots s_k = t s_1 \dots \hat{s_i} \dots s_k$. Now $s_1 = s \ne t$, so we cannot have that $i = 1$. Hence, w is represented by a word starting with $(t, s, \dots)$.

For $q \ge 2$, let $\underline{s}_q$ be $(\dots s, t, s)$ the length q alternating word with last letter s. We will show by induction on q that for any $q \le m_{st}$, we can find a reduced word for w beginning with $\underline{s}_q$. Then because w has finite length, $\Rightarrow$ m_{st} is finite and the case $q = m_{st}$ proves the claim.

Base case: $q=1$ done: $w = s s_2 \dots s_k$ and $(s, s_2, \dots, s_k)$ is reduced.

Ind. hyp: we have a reduced word $\underline{s}'$ representing w that begins with $\underline{s}_{q-1}$.

Let $s' = \begin{cases} s & \text{if } q-1 \text{ even, i.e. } \underline{s}_{q-1} \text{ starts in } t \\ t & \text{if } q-1 \text{ odd, i.e. } \underline{s}_{q-1} \text{ starts in } s. \end{cases}$

Then $\ell_S(s'w) < \ell_S(w)$ (remember we have reduced words $(s, s_2, \dots, s_k)$ and $(t, t_2, \dots, t_k)$ for w, so this is just true regardless of odd/even). Hence by (E), we can find another reduced word for w by exchanging a letter u of $\underline{s}'$ for an s' at the start.

Suppose u in $\underline{s}_{q-1} \subset \underline{s}'$, i.e. u is one of the first $q-1$ letters of $\underline{s}'$. Then it follows there are two distinct reduced words representing $\underline{s}_{q-1}$ using only letters s and t. I think this follows by playing the game above with word $\underline{s}_{q-1}$. Perhaps instead you just get that, because no other parts of w changes and $\underline{s}_{q-1} = (\dots, s, t)$ is reduced, then this other word for $\underline{s}_{q-1}$ must also be reduced. However, $q-1 < m_{st}$ by hypothesis, and in $W_{\{s,t\}}$, the only braid relation is $\underbrace{stst\dots}_{m_{st}} = \underbrace{tsts\dots}_{m_{st}}$ (remember $(st)^{m_{st}} = e$).

So any reduced expression of less than length m is unique, by observing paths in $\mathrm{Cay}_{\{s,t\}}(W_{\{s,t\}})$.

So we cannot have u in $\underline{s}_{q-1}$. After applying (E) like we said above, we get a reduced word for W starting with $s'\,\underline{s}_{q-1} = \underline{s}_q$.

Therefore we have completed the induction on $q \le m_{st}$, hence setting $q = m_{st}$ gives $\underline{u}$ if m_{st} is odd or $\underline{u}'$ if m_{st} is even. This completes the proof of (2).

($\Leftarrow$) is trivial. Braid moves dont affect the element the word represents $\square$

proof of (1):

($\Rightarrow$) if a word is reduced, it cannot be shortened at all. PERIODT!

($\Leftarrow$) Suppose $\underline{s} = (s_1, \dots, s_k)$ cannot be shortened by a sequence of deleting (s,s) pairs and braid moves. We show by induction on k that $\underline{s}$ is reduced.

Base: $k = 1$ ✓ so let $k > 1$.

Ind. hyp: Suppose true $\forall$ words of length $k-1$.

Standard argument, and $\Rightarrow$ $\ell(\underline{s}') = k-1$.

we'll show then it can be shortened by a finite sequence of Braid moves + (s,s) deletion.

Then $\underline{s}' = (s_2, \dots, s_k)$ is reduced for s, w. Suppose $\underline{s}$ is not reduced. Let $w = s_1 \dots s_k$ and $w' = s_2 \dots s_k$.

Then $\underbrace{\ell_S(s, w')}_{=} = \ell_S(w) \le k-1$

$\ell_S(w)$ is not reduced so $\ell_S(w) < k$.

By (E), $w' = s, s_2 \dots \hat{s}_i \dots s_k$ and $s'' = (s_1, s_2, \dots, \hat{s}_i, \dots, s_k)$ has length $k-1$ and so is reduced. By part (2) of the Theorem, s' and s'' are both reduced words for $w' = s_2 \dots s_k$ of length $k-1$, and therefore by induction s' and s'' are related by a finite sequence of Braid moves

Hence $\underline{s}$ can be transformed into a word starting with (s_1, s_1) by a finite sequence of Braid moves, so $\Rightarrow$ s can be shortened by a finite sequence of deleting (s,s) pairs and Braid moves

Proof of Theorem 3.1 (4) $\Rightarrow$ (1): *the exchange condition* $\Rightarrow$ (W,S) is a coxeter system.

Suppose W is a group generated by a distinct set of involutions $S = \{s_i\}_{i\in I}$. Assume (E) holds. We want to show that (W,S) is a coxeter system.

Let m_{ij} be the order of $s_i s_j$ in W. Define a coxeter system using the matrix (m_{ij}) : (W', S') generators $S' = \{s_i'\}_{i\in I}$.

Then $\phi : W' \to W$; $s_i' \to s_i$ is a surjective homomorphism by the universal property of presentation of W'.

We want to show that ϕ is injective : $\Rightarrow$ $W' \cong W$, so (W,S) is a coxeter system.

Suppose that $w' \in \ker(\phi)$ and $w' \neq e$. Then w' is represented by a reduced word $(s_1', \dots, s_k')$ in S', so $\phi(w)$ is represented by $(s_1, \dots, s_k)$ in S. Since $\phi(w) = e$, $\Rightarrow$ $(s_1, \dots, s_k)$ cannot be reduced. By Tit's thm, $\Rightarrow$ $(s_1, \dots, s_k)$ can be shortened by a finite sequence of Braid moves and deleting (s,s) subwords. But then $\Rightarrow$ $(s_1', \dots, s_k')$ is not reduced. $\lightning$.

$\Rightarrow$ ϕ is injective and hence (W,S) is a coxeter system. $\square$

4. Tit's representation

Thm (Tit's): Let I be a finite indexing set, and let $S = \{s_i\}_{i \in I}$, and let $M = \{m_{ij}\}_{i,j \in I}$ be a coxeter matrix. Then there's a faithful representation $\rho: W \to GL_n(\mathbb{R})$, where $W = \langle S \mid (s_i s_j)^{m_{ij}} = e \rangle$, where $n = |S| = |I|$, and such that

- $\forall i$, $\rho(s_i) =: \sigma_i$ is a linear involution with fixed point set a hyperplane
- for all i,j, the product $\sigma_i \sigma_j$ has order m_{ij}.

The homomorphism $\rho: W \to GL(n, \mathbb{R})$ is sometimes known as the *canonical representation.*

N.B. $\sigma_i \in GL_n(\mathbb{R})$ won't usually be an orthonormal reflection.

Construction of the Tits representation : let (W,S) be as above. Wlog $I = \{1, \dots, n\}$. Let $V =$ n-dimensional vector space with basis $e_1, \dots, e_n$. Define a symmetric bilinear form B on V as follows:

$$B(e_i, e_j) = \begin{cases} -\cos(\pi/m_{ij}) & \text{if } m_{ij} \text{ finite} \\ -1 & \text{if } m_{ij} \text{ infinite} \end{cases}$$

Note $B(e_i, e_i) = 1$ and $B(e_i, e_j) \leq 0$ for $i \neq j$.

Define $\sigma_i : V \to V$ by $\sigma_i(v) = v - 2B(e_i, v)e_i$ looks like reflecting in e_i

First properties:

- σ_i is a linear map
- $\sigma_i(e_i) = -e_i$
- Fixed points of σ_i : $\mathrm{Fix}(\sigma_i) = \{v \in V : B(e_i, v) = 0\} =: H_i$ hyperplane (dim $n-1$)
- σ_i preserves the bilinear form : $B(\sigma_i(e_j), \sigma_i(e_k)) = B(e_j, e_k)$.

$$\left.\begin{aligned} B(\sigma_i(e_j), \sigma_i(e_k)) &= B(e_j - 2B(e_i,e_j)e_i,\ e_k - 2B(e_i,e_k)e_i) \\ &= B(e_j, e_k) + B(e_j, -2B(e_i,e_k)e_i) \\ &\quad + B(-2B(e_i,e_j)e_i, e_k) + B(-2B(e_i,e_j)e_i, -2B(e_i,e_k)e_i) \\ &= B(e_j, e_k) + 2\cos(\pi/m_{ik})(-\cos(\pi/m_{ij})) \\ &\quad + 2\cos(\pi/m_{ij})(-\cos(\pi/m_{ik})) + 4\cos(\pi/m_{ij})\cos(\pi/m_{ik}) B(e_i, e_i) \\ &= B(e_j, e_k) - 4\cos\left(\frac{\pi}{m_{ij}}\right)\cos\left(\frac{\pi}{m_{ik}}\right) + 4\cos\left(\frac{\pi}{m_{ij}}\right)\cos\left(\frac{\pi}{m_{ik}}\right) = B(e_j, e_k). \end{aligned}\right\} \text{respects bilinear form}$$

$$\left.\begin{aligned} \sigma_i^2(v) &= \sigma_i(\sigma_i(v)) \\ &= \sigma_i(v - 2B(e_i,v)e_i) \\ &= v - 2B(e_i,v)e_i - 2B(e_i, v - 2B(e_i,v)e_i)e_i \\ &= v - 2B(e_i,v)e_i - 2B(e_i,v)e_i + 4B(e_i,v)B(e_i,e_i)e_i \\ &= v - 4B(e_i,v)e_i + 4B(e_i,v)e_i = v. \end{aligned}\right\} \text{involution}$$

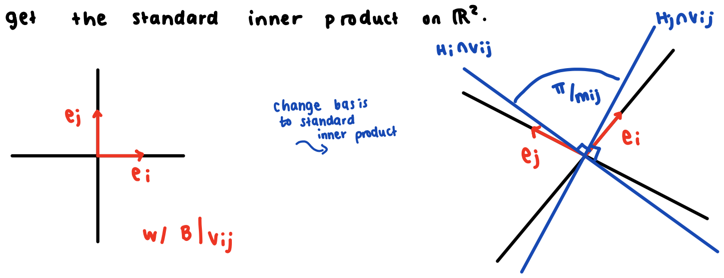
Proposition 4.2: $\sigma_i \sigma_j$ has order m_{ij} for all $i,j \in I$.

Corollary 4.3: The map $s_i \mapsto \sigma_i$ extends to a homomorphism $\rho: W \to GL_n(\mathbb{R})$

proof of 4.2: • if $i=j$, we're done. (an involution)

• Assume $i \neq j$. Let $V_{ij} = \text{span}(e_i, e_j)$. Then $\sigma_i(V_{ij}) = V_{ij} = \sigma_j(V_{ij})$. So consider the restriction of $\sigma_i \sigma_j$ to V_{ij}.

Case a) m_{ij} finite: The matrix repn of $B|_{V_{ij}}$ wrt (e_i, e_j) $= \begin{pmatrix} 1 & -\cos(\pi/m_{ij}) \\ -\cos(\pi/m_{ij}) & 1 \end{pmatrix}$ has det >0 and tr >0 and so is positive definite. So after a change of basis, we get the standard inner product on $\mathbb{R}^2$.



angle between e_i and e_j

$-\cos(\pi/m_{ij}) = \cos(\pi - \pi/m_{ij})$

$\Rightarrow$ angle $= \pi - \pi/m_{ij}$

So $\sigma_i|_{V_{ij}}$ = the orthonormal reflection in H_i and similarly σ_j (after change of basis)

Upshot: $\sigma_i \sigma_j|_{V_{ij}}$ is a rotation by angle $2\pi/m_{ij}$ ($\Rightarrow$ of order m_{ij}). on V_{ij}

Note that $V_{ij}^{\perp} := \{ w \in V : B(w,v) = 0 \ \forall v \in V_{ij} \}$, $V \cong V_{ij} \oplus V_{ij}^{\perp}$ (direct since $B|_{V_{ij}}$ is +ve def (non degenerate)).

But $\sigma_i \sigma_j|_{V_{ij}^{\perp}} = \text{Id}$, hence $\sigma_i \sigma_j$ has order m_{ij} on V, as required.

Summary of idea of proof: Can think about how σ_i and σ_j act on e_i and e_j. On the orthog. complement of $\langle e_i, e_j \rangle$, σ_i and σ_j act trivially. And we can represent $\sigma_i \sigma_j$ as a rotation, with angle $\frac{2\pi}{m_{ij}}$, which has order m_{ij}.

Case b) m_{ij} infinite: matrix repn of $B|_{V_{ij}}$ wrt (e_i, e_j) $= \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is positive semi definite, but not definite.

Calculate $\sigma_i \sigma_j(e_i) = \sigma_i(e_i + 2e_j) = e_i + 2(e_i + e_j) \Rightarrow (\sigma_i \sigma_j)^k(e_i) = e_i + 2k(e_i + e_j)$

which clearly has infinite order. So we're done. $\square$

Corollary 4.4: Let (W,S) be a Coxeter system. Then elements of S are pairwise distinct.

proof: $\sigma_i \neq \sigma_j$ (use $\sigma_i \sigma_j$ has order m_{ij}, or notice they do different things to e_i, say. Different linear maps) $\square$

| They're distinct in the representation, so distinct preimages.

Corollary 4.5: $s_i s_j$ has order m_{ij} in W

proof: Immediate as $\sigma_i \sigma_j$ has order m_{ij}.

$\square$

Geometry when $m_{ij} = \infty$

Matrix repn is $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. We have $\text{Null}(B|_{V_{ij}}) = \langle e_i + e_j \rangle =: N$. Taking the quotient by null space, $B|_{V_{ij}}$ induces a +ve def. form on V_{ij}/N. ← 1 dimensional

Notation: $W_{ij} = \langle s_i s_j \rangle \leq W$, $W_{ij} \cong D_\infty$ we'll recover action from before.

The matrix representation of B when restricted to V_{ij} is given by $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ (in basis $\{e_i, e_j\}$). Hence B induces a +ve definite form on B/N, which is one dimensional.

Let $W_{ij} : \langle s_i, s_j \rangle \simeq D_\infty$. Then W_{ij} (via ρ) has the following properties:

1) W_{ij} acts faithfully on V_{ij}

faithful := if $g \cdot x = x \ \forall x \in X$, then $g = e$.

2) We have $\sigma_i(e_i + e_j) = \sigma_j(e_j + e_i) = e_i + e_j$

$\Rightarrow \sigma_i, \sigma_j$ fix N pointwise.

Note: $H_i \cap V_{ij} = N$, $H_j \cap V_{ij} = N$, so not a very fruitful viewpoint.

$H_i := \{ v \in V : B(e_i, v) = 0 \}$

On V_{ij}, B is represented by the matrix $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $\underbrace{\ker(B)}_{= \mathbb{N}?} = H_i \cap V_{ij} = H_j \cap V_{ij}$.

Idea: Consider dual vector space $V_{ij}^* := \mathcal{L}(V_{ij}, \mathbb{R})$. We have a dual representation $\rho^* : W_{ij} \to GL(V_{ij}^*)$

$w \cdot \varphi := (w, \varphi)(v) = \varphi(w^{-1} v)$, where $w \in W_{ij}$, $\varphi \in V_{ij}^*$, $v \in V_{ij}$.

$W_{ij} = \langle s_i, s_j \rangle \leq W$

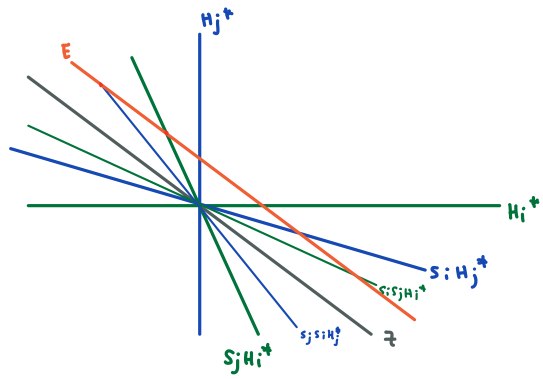
This is faithful as it's dual is faithful.

want w to give us a GL map on V_{ij}^*, i.e. one A where we act on V_{ij}^* and get a map $V_{ij} \to \mathbb{R}$.

Note that $V_{ij}^* = (\mathrm{span}(e_i, e_j))^* \circlearrowleft W_{ij} = \langle s_i, s_j \rangle \simeq D_\infty$.

$V_{ij}^* = (\mathrm{span}(e_i, e_j))^* \circlearrowleft W_{ij}$ by the above representation, $W_{ij} \cdot V_{ij}^*$; $(w \cdot \varphi)(v) = \varphi(w^{-1} v)$ which gives another map $w \cdot \varphi \in V_{ij}^*$.

Denote by $H_i^* = \{ \varphi \mid \varphi(e_i) = 0 \}$, and let
$Z : \{ \varphi \mid \varphi(e_i + e_j) = 0 \}$ $(= (V_{ij}/N)^*)$



Since W_{ij} fixes $e_i + e_j$, the action of W_{ij} on V_{ij}^* preserves Z.

Calculating $\rho^*(s_i) = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ wrt. obvious basis.
$\rho^*(s_j) = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$ } not sure where this comes from.

Let $E = Z + 1$

E has standard action of D_∞ acting on it.

intersection points of orange line with green / blue lines are supposed to be equidistant.

Have $\rho^*: W_{ij} \longrightarrow GL(V_{ij}^*)$; $(\rho^*(w)(\varphi))(v) = \varphi(\rho(w^{-1})(v))$

For s_i, $s_i^2 = e$ so $s_i^{-1} = s_i$. Let $\varphi_i : v \longmapsto B(e_i, v)$. Then

$$\rho^*(s_i)(\varphi)(v) = \varphi(\rho(s_i^{-1}))(v) = \varphi \circ (\sigma_i)(v) = \varphi(v - 2B(e_i,v)e_i)$$
$$= \varphi(v) - 2\varphi(e_i)\varphi_i(v)$$

with the obvious basis φ_i, φ_j,

$$\rho^*(s_i)(\varphi_i)(v) = \varphi_i(v) - 2\underbrace{\varphi_i(e_i)}^{=1}\varphi_i(v)$$
$$= -\varphi_i(v) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\rho^*(s_i)(\varphi_j)(v) = \varphi_j(v) - 2\underbrace{\varphi_j(e_i)}_{=-1}\varphi_i(v)$$
$$= \varphi_j(v) + 2\varphi_i(v) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

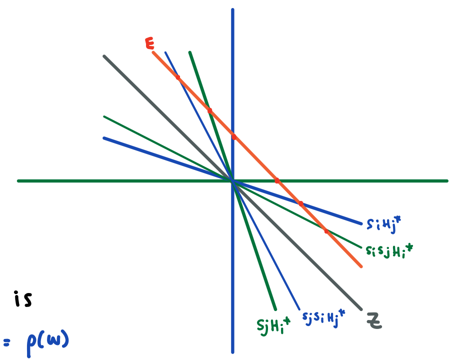
gives us $\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$

not sure where I am going wrong.

$$\rho^*(s_j)(\varphi_i)(v) = \varphi_i(v) - 2\varphi_i(e_j)\varphi_j(v) \quad \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
$$= \varphi_i + 2\varphi_j(v)$$

$$\rho^*(s_j)(\varphi_j)(v) = \varphi_j(v) - 2\varphi_j(e_j)\varphi_j(v)$$
$$= \varphi_j(v) - 2\varphi_j(v)$$
$$= -\varphi_j(v) \quad \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$



Faithfulness of Tit's representation: Dual representation

$\rho^*: W \to GL(V^*)$ given by $(\rho^*(w)(\varphi))(v) = \varphi(\rho(w^{-1}))(v)$, $w \in W$, $\varphi \in V^*$, $v \in V$

Goal: ρ^* faithful ($\Leftrightarrow \rho$ is too)

Define $\varphi_i \in V^*$ by $\varphi_i(v) = B(e_i, v)$. Then $\sigma_i^* := \rho^*(s_i)$ is
$\sigma_i^*(\varphi) = \varphi - 2\varphi(e_i)\varphi_i$

remember σ_i an involution so $\rho(w^{-1}) = \rho(w)$

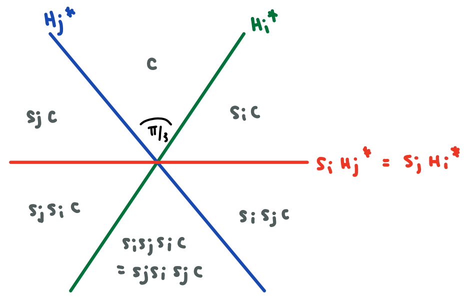
Remember the hyperplane $H_i^+ := \{\varphi \in V^* \mid \varphi(e_i) = 0\}$, and define the (open) halfspace $C_i = \{\varphi \in V^* \mid \varphi(e_i) > 0\}$,
and $C := \bigcap_{i \in I} C_i$, closure $\bar{C} :=$ chamber associated to representation. Finally denote $C_{ij} = C_i \cap C_j$.

Recall $\sigma_i(v) = v - 2B(e_i, v)e_i$

$$\sigma_i^*(\varphi)(v) = \varphi(v) - 2\varphi(e_i)\underbrace{\varphi_i(v)}_{B(e_i,v)}$$
$$= \varphi(v) - 2B(e_i,v)\varphi(e_i)$$
$$= \varphi(v - 2B(e_i,v)e_i)$$
$$= \varphi(\sigma_i(v))$$
$$= \varphi(\sigma_i^{-1}(v)) \quad \text{as required.}$$

Example: m_{ij} finite : $\mathrm{span}(\varphi_i, \varphi_j) = \mathbb{E}^2$ with standard inner product.

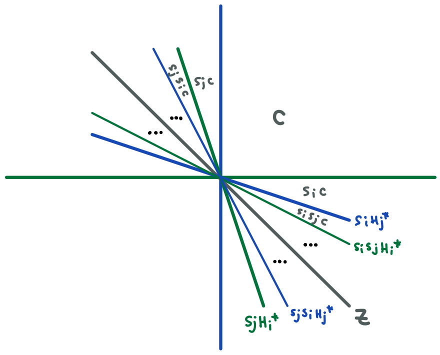
e.g. if $m_{ij} = 3$



$c = c_{ij}$

On $\bigcap_{k \notin j,i} H_k^*$, $\circlearrowleft$ s_i s_j fix pointwise.

m_{ij} infinite : 2d containing φ_i, φ_j.



$c = c_{ij}$

$\bigcap_{k \neq 0} H_k^*$ $\circlearrowleft$ s_i, s_j trivial

> This reads a little funny. Basically :
> finite: ① on $\bigcap_{k \neq i,j} H_k^*$, the two elements s_i and s_j fix the space pointwise
> infinite: ② on $\bigcap_{k \neq 0} H_k^*$, s_i and s_j act trivially (preserves the space)

Definition: Let G be a group acting on a set Π. $\mathcal{C} \subset \Pi$ is prefundamental for G if $\forall g \in G$, $g\mathcal{C} \cap \mathcal{C} \neq \emptyset \Rightarrow g = 1 \in G$

Ex : C_i is a prefundamental for $W_i := \langle s_i \rangle$ acting on V^*

C_{ij} is prefundamental for W_{ij}.

> This all feels intuitive but its important to know whats going on. $C_i := \{ \varphi \in V^* : \varphi(e_i) > 0 \}$. For s_i acting on C_i, $(s_i \cdot \varphi)(e_i) = \varphi(\rho(s_i^{-1})e_i) = \varphi(\sigma_i(e_i)) = \varphi(-e_i) = -\varphi(e_i) < 0$. And since $s_i^2 = e$, $\Rightarrow \forall w \in W_i$, $w\mathcal{C} \cap \mathcal{C} \neq \emptyset \Rightarrow w = e$. So C_i is prefundamental for w_i.
>
> Similar logic shows the second statement.

Theorem 4.6 (Tits) The data $(W, S, \{C_i\})$ satisfies property P: for any $w \in W$, $i \in I$, either $wC \subset C_i$, or $wC \subset s_i C_i$. Moreover, in second case, $\ell_S(s_i w) = \ell_S(w) - 1$.

Corollary 4.7 (ES2) C is prefundamental for W $\Rightarrow$ ρ^* is faithful

Key : we already have that $(W_{ij}, \{i,j\}, \{c_i, c_j\})$ satisfies property P. from our pictures

Strategy: P_n = (P true for all w with $\ell(w) = n$)
Q_n = ($\forall w \in W$ with $\ell(w) = n$, $i \neq j$, $\exists\, \mu \in W_{ij}$ s.t $wC \subset \mu C_{ij}$ and $\ell(\mu^{-1}w) = \ell(w) - \underbrace{\ell'(\mu)}_{\text{wrt } s_i, s_j}$)

proof of theorem: P_0 and Q_0 hold ✓ . Want to show a) (P_n and Q_n) $\Rightarrow$ P_{n+1}, and b) (P_{n+1} and Q_n) $\Rightarrow$ Q_{n+1}.

a) Suppose $\ell(w) = n+1$, and $s_i \in S$. Then $w = s_j w'$ for some $s_j \in S$, $\ell(w') = n$.

- if $i = j$: apply P_n to w'. Must have $w'C \subset C_i$ $\Rightarrow$ $s_i w' C \subset s_i C_i$ and $\ell(sw) = n$ ✓.

> If $w'C \subset s_i C_i$ instead, then $\ell(s_i w') = \ell_s(w') - 1 \Leftrightarrow \ell(w) = n-1$ ↯

- if $i \neq j$: apply Q_n to w'. Then $\exists\, \mu \in W_{ij}$ s.t $w'C \subset \mu C_{ij}$ and $\ell(\mu^{-1}w') = \ell(w') - \ell'(\mu)$. Two possibilities:

(i) $s_j \mu C_{ij} \subset C_i$ $\Rightarrow$ $wC \subset C_i$, or
(ii) $s_j \mu C_{ij} \subset s_i C_i$ $\Rightarrow$ $wC \subset s_i C_i$.

> Remember μ is a word in s_i and s_j, and definitely $C_{ij} \subset C_i$ and $\subset C_j$ will have $s_j \mu C_{ij} \subset C_i$ or $s_i C_i$ because $C_{ij} \subset C_i$ and C_i and $s_i C_i$ are separated only by the half space H_i.
>
> Now if $s_j(\mu C_{ij}) \subset C_i$, $s_j(w'C) \subset s_j(\mu C_{ij}) \subset C_i \Rightarrow wC \subset C_i$. Similarly for second possibility

Word length for (ii)? $\ell(s_i w) = \ell(s_i s_j w') \leq \ell(s_i s_j \mu) + \ell(\mu^{-1} w')$.

$$\leq \underbrace{\ell'(s_j\mu) - 1}_{3} + \underbrace{\ell(w') - \ell(\mu)}_{4} \leq \ell(w) - 1$$

must be equal (cant differ by more than 1)

> If μ starts with s_j, then $\ell'(s_j\mu) \leq \ell(\mu)$, and we get that $* \leq \ell(w') - 2 = \ell(w) - 1$
> if μ starts with s_i, then $\ell(s_j\mu) = \ell(\mu) + 1$, and so $* = \ell(w') = \ell(w) - 1$.
> So in total we have $\ell(s_i w) \leq \ell(w) - 1$. But $\ell(s_i w)$ cannot differ from $\ell(w)$ by more than 1. So $\ell(s_i w) = \ell(w) - 1$.

b) Suppose $\ell(w) = n+1$, $i \neq j$. If $wC \subset C_{ij}$, then done ($\mu = 1$). Assume not. Wlog $wC \not\subset C_i$. Apply P_{n+1}, $wC \subset s_i C_i$, and $\ell(s_i w) = \ell(w) - 1$. Apply Q_n to $s_i w$, so $\exists\, v \in W_{ij}$ s.t. $s_i w C \subset v C_{ij}$ and $\ell(s_i w) = \ell'(v) + \ell(v^{-1} s_i w)$. Then $wC \subset s_i v C_{ij}$ and $\ell(w) = 1 + \ell(s_i w) = 1 + \ell'(v) + \ell(v^{-1} s_i w)$

$$\geq \ell'(s_i v) + \ell((s_i v)^{-1} w) \geq \ell(w)$$

Both of these then must be equalities so that $\ell((s_i v)^{-1} w) = \ell(w) - \ell'(s_i v)$.

Change in notation: replace C with C°, $\bar{C}$ with C to agree with notation in literature.

4.9. Definition: the Tits cone of (W, S) is $\bigcup_{w \in W} wC \subset V^*$

4.10 Example

1) D_{2n} n finite, then $V^* \leftrightarrow \mathbb{E}^2$, and the Tits cone is all of $\mathbb{E}^2$.

remember taking closure

2) D_∞ : $V^* = V_{ij}^*$. Tits cone is $\{\varphi \in V_{ij}^* \mid \varphi(e_i + e_j) > 0\} \cup \{0\}$
and the interior is the open half space bounded by z and containing z.

> Can see this from pictures.

5. Finite Coxeter Groups

5.1 Definition: Let (W,S) be a Coxeter system. Then (W,S) is reducible if $S = S' \sqcup S''$ such that $m_{ij} = 2$ $\forall\, s_i \in S'$, $s_j \in S''$, i.e. $s_i s_j = s_j s_i$ is a rel in W $\forall\, s_i \in S'$, $s_j \in S''$.

(W,S) is irreducible if its not reducible.

5.2. Remark: if (W,S) reducible, then $W \cong \langle S \rangle \times \langle S' \rangle$
But W can be irreducible and still split as a product. e.g. $D_{2(2k)} \cong D_{2k} \times C_2$.

5.3. Theorem: Let (W,S) be irreducible and $|S| = n$. Then the following are equivalent.

i) W is a geometric reflection group on S^{n-1} generated by $S = \{s_i\}_{i \in I}$, and the set of reflections in codimension 1 are faces $\{F_i\}_{i \in I}$ of a simplex in S^{n-1} s.t. F_i and F_j meet at an angle π / m_{ij}.

(ii) B is positive definite

(iii) W is finite.

proof: in Davis section 6: uses Thm 1.11.

As an aside, we have similar theorems for Euclidean (B positive-semidefinite of corank 1), and Hyperbolic.

If W is a finite Coxeter group with $|S| = n$, then $V^* \leftrightarrow \mathbb{E}^n$, and C (formerly $\bar{C}$) is a closed Euclidean simplicial cone with boundary given by hyperplanes.

C° is prefundamental for W.

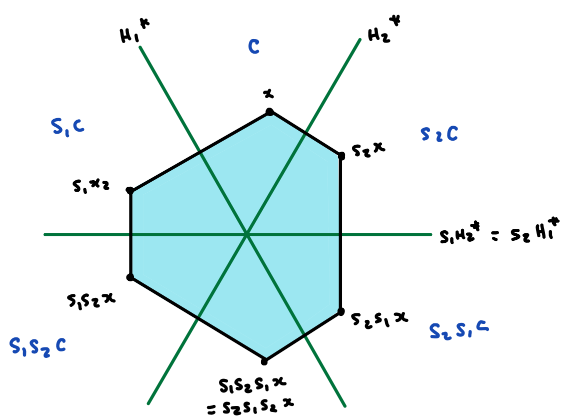
Recall Corollary 4.7 (ES2) which says $w \in W$ then if $wC^\circ \cap C^\circ \neq \emptyset$, then $w = e$. This implies if $x \in C^\circ$, then the orbit Wx has $|W|$ points (they all have to be different)

5.4. Definition Let (W,S) be finite. The Coxeter polytope for W is the convex hull of the W orbit on V^* of a point $x \in C^\circ$.

These are convex Euclidean polytopes but are not in general regular

if say $wx = vx$, then
$v^{-1}wx = x \Rightarrow$
$v^{-1}wx \in v^{-1}wC^\circ \cap C^\circ \neq \emptyset$
so $v^{-1}w = e \Rightarrow v = w$.

5.5 Example D_6



Rem: the 1-skeleton is isomorphic as a nonmetric graph to $\mathrm{Cay}_S(W)$.

Forms B associated to irreducible coxeter systems can be classified by graphs. This lead to Coxeter's classification of finite coxeter groups.

5.6. Definition

A **coxeter - Dynkin diagram** Γ is a simple labelled graph with finite vertex set $V(\Gamma) = S = \{s_i\}_{i \in I}$ and edge labels $s_i \overset{m_{ij}}{—\!—\!—} s_j$ where $m_{ij} \geq 3$ or $m_{ij} = \infty$

5.7 Lemma: There is a 1-1 correspondence between coxeter system (W,S) and Coxeter - Dynkin diagrams Γ.

proof: We give a bijection

$$(W,S) \text{ coxeter system} \longleftrightarrow \Gamma \text{ coxeter diagram}$$

$$S \longleftrightarrow V(\Gamma)$$

$$m_{ij} = m_{ji} = 2 \longleftrightarrow \text{no edge between } \overset{\bullet}{s_i} \quad \overset{\bullet}{s_j}$$

$$m_{ij} \geq 3, = \infty \longleftrightarrow s_i \overset{m_{ij}}{—\!—\!—} s_j$$

5.8 Notation: we omit edge labels $m_{ij} = 3$ for rest of the course, i.e.

$s_i \overset{3}{—\!—\!—} s_j$ is just represented by $s_i —\!—\!— s_j$

Under the above bijection, denote image of Γ by $(W(\Gamma), V(\Gamma))$, or $(W(\Gamma), S)$.

5.9. Remark: many mathematicians use a different convention where $\overset{\bullet}{s_i} \quad \overset{\bullet}{s_j} \iff m_{ij} = \infty$

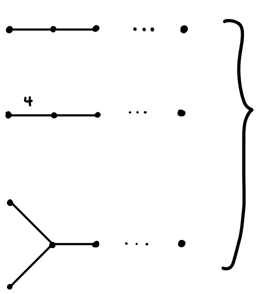
5.10. Theorem (Coxeter 1930s) (Classification of finite Coxeter groups)

(W,S) gives rise to a finite Coxeter group $W \iff (W,S) = (W(\Gamma), V(\Gamma))$ for Γ a disjoint union of a finite number of the following graphs.

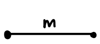
$A_n \ (n \geq 1)$

$B_n \ (n \geq 2)$

$D_n \ (n \geq 4)$



n vertices

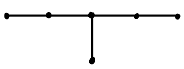
$I_2(m)$  $m \geq 5$

F_4

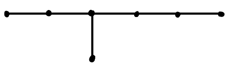
H_3

H_4

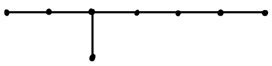
E_6



E_7



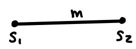
E_8



5.11 Remark : $\Gamma = \Gamma_1 \cup \Gamma_2$ precisely when $(W(\Gamma), V(\Gamma))$ is reducible.

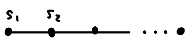
5.12 Examples:

D_{2m} : $W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^m = e \rangle$



$m = 3$ $W(A_2)$
$m = 4$ $W(B_2)$
$m \geq 5$ $W(I_2(m))$

A_{n-1} $(n \geq 1)$



$n-1$ vertices, then $W(\Gamma) \cong S_{n-1}$.

check indices

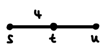
$W(A_{n-1}) = \langle s_1, \dots, s_{n-1} \mid s_i^2 = e,\ (s_i s_{i+1})^3 = e,\ (s_i s_j)^2 = e$ otherwise $(i - j \geq 2) \rangle$

Given a Coxeter diagram Γ, let $S = V(\Gamma)$, and let Γ_T be the full subgraph of Γ spanned by a subset of the vertices $T \subseteq S$.

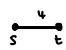
Full subgraph: if $t_1, t_2 \in T$, and in Γ $\exists$ an edge $t_1 - t_2$, then in Γ_T we have the same labelled edge (induced subgraph)

Then $(W(\Gamma_T), T)$ is a coxeter system

e.g. $\Gamma =$



, $T = \{s, t\}$, then $\Gamma_T =$



.

5.13: Definition: if you take (W, S) a Coxeter system, $T \subseteq S$, then the parabolic Subgroup W_T of W is $W_T = \langle T \rangle$. If $T = \emptyset$, then fix $W_\emptyset = \{e\}$ (the trivial group).

5.14 Lemma: if (W, S) is a Coxeter system, and W_T, $(W(\Gamma_T), T)$ as defined above for some subset $T \subseteq S$, then $W(\Gamma_T) \cong W_T$.

proof: if $|S| = n$, and V be an n-dimensional vector space with basis e_s, $s \in S$. Then let $\rho: W \to GL(V)$ be the Tits representation with symmetric bilinear form B. Let G_T be the subgroup of $GL(V)$ which stabilizes (as a subspace) $V_T := \operatorname{span}\{e_t : t \in T\}$ (not elementwise)

Now $(W(\Gamma_T), T)$ has it's own Tits representation of the form B_T with vector space $V' := \langle e_t' \mid t \in T \rangle$.
Then $V' \longrightarrow V$; $e'_t \mapsto e_t$ is a vector space inclusion.

think of matrix

By naturality of the Tits representation (i.e. $B|_T = B_T$) we get a commutative diagram

$$W(\Gamma_T) \xhookrightarrow[4.7]{\rho} GL(V')$$

universal property of a group

restricting V to $V_T \cong V'$

presentation

$$W_T \xrightarrow{\varphi|_{V_T}} G_T$$

(ρ^* is faithful) → other restrictions are just injections.

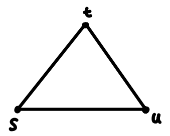
Top arrow injection by Cor 4.7 $\Rightarrow$ left arrow is an injection, which gives $W(\Gamma_T) \cong W_T$ as required. $\square$

5.15 Definition: if a parabolic subgroup is finite, we call it a spherical subgroup

5.16 Corollary: Combining Theorem 5.10 with lemma 5.14, we see that all spherical subgroups can be obtained by observing Γ for (W,S)

5.17. Example: $(3,3,3)$ - triangle group

Coxeter graph



$S = \{s,t,u\}$

(W,S) has spherical subgroups:

$W_\emptyset$,
W_s, W_t, W_u type A_1
$W_{\{s,t\}}$, $W_{\{t,u\}}$, $W_{\{s,u\}}$ of type A_2

5.18 Theorem: (W,S) Coxeter System. Then

(a) (W_T, T) is also a coxeter system $\forall\, T \subseteq S$.

(b) For all $T \subseteq S$, $w \in W_T$, $\ell_T(w) = \ell_S(w)$, and any reduced word for w in S, $s_1 \cdots s_k$ satisfies $s_i \in T$ $\forall i$.

(c) if $T, T' \subseteq S$, then $W_T \cap W_{T'} = W_{T \cap T'}$, and $\langle W_T, W_{T'} \rangle = W_{T \cup T'}$.

(d) The bijection $T \to W_T$; $\{$ subsets $T \subseteq S\} \longrightarrow \{$ parabolic subgroups of $W\}$ preserves the partial ordering on both sets given by inclusion.

5.19 : Lemma : For (W,S) a Coxeter system, $w \in W$, then $\exists$ subset $S(w) \subseteq S$ such that given any reduced word $(s_1 \cdots s_k)$ representing w, $S(w) = \{s_1, \ldots, s_k\}$

i.e. $S(w)$ depends only on the element w and not its word representation.

proof : by contradiction: Let w be a minimal length counterexample, i.e. $w = s_1 \cdots s_k = t_1 \cdots t_k$ such that $s_i, t_i \in S$ and $\{s_1, \ldots, s_k\} \neq \{t_1, \ldots, t_k\}$. Then $w = s_1 v$ where $(s_2, \ldots, s_k)$ is also reduced for v. By the exchange condition, $\ell_S(s_1 w) < \ell(w)$ so $\exists\, i$ s.t. $w = s_1 t_1 \cdots \hat{t_i} \cdots t_k$. So v satisfies $S(v) \subset \{t_1, \ldots, t_k\}$

Since $\ell_S(v) < \ell_S(w)$, it follows that $\{s_2, \ldots, s_k\} = S(v) \subset \{t_1, \ldots, t_k\}$ by the assumption of a minimal length counterexample.

By same argument on $w^{-1} = s_k \cdots s_1$, we get $\{s_{k-1}, \ldots, s_1\} \subset \{t_1, \ldots, t_k\}$, so $\{s_1, \ldots, s_k\} \subseteq \{t_1, \ldots, t_k\}$. By symmetry of argument, $\{t_1, \ldots, t_k\} \subseteq \{s_1, \ldots, s_k\}$, so $\{s_1, \ldots, s_k\} = \{t_1, \ldots, t_k\}$, which is a contradiction by assumption of a minimal length counterexample. $\square$

Proof of Theorem 5.18:

(a) follows from Lemma 5.14 ($W_T \cong W(\Gamma_T)$)

deletion condition

(b) Use lemma 5.19. If $w \in W_T$, then $S(w) \subset T$. So by lemma, it follows that if $(s_1, \dots, s_k)$ is a reduced word for w, then each $s_i \in T$. So $\ell_S(w) = \ell_T(w)$.

(c) Clearly $W_{T \cap T'} \subset W_T \cap W_{T'}$. For reverse inclusion, $W_T \cap W_{T'} \subset W_{T \cap T'}$: Then by lemma 5.19, if $w \in W_T \cap W_{T'}$, then $s(w) \subset T$ and $s(w) \subset T'$, so $s(w) \subset T \cap T'$ ($s(w)$ is unique) and so $w \in W_{T \cap T'}$.
The second part $\langle W_T, W_{T'} \rangle = W_{T \cup T'}$ is an exercise.

(d) Suffices to show if $T' \subset T$, then $W_{T'} \subset W_T$ (strict). Then from c), we can take
$$W_{T \cap T'} = W_{T'} = W_{T'} \cap W_T$$
So $W_{T'} \subseteq W_T$. Let $s \in T$, $s \notin T'$. By lemma 5.19, $s(s) = \{s\}$, so any reduced word representing s only involves s, which $\notin T'$. So $s \notin W_{T'}$, but $s \in W_T \Rightarrow W_{T'} \subset W_T$ (strict)

5.20 Definition: Given a cox. system (W, S), let $\mathcal{S} = \{ T \subseteq S : W_T \text{ is spherical} \}$

5.21 Remark: $\mathcal{S}$ depends on (W, S), but this is not reflected in the notation.

6. The Basic Construction

6.1 Definition: An (abstract) simplicial complex is a (possibly infinite) set V - the vertex set, and a collection X of finite subsets of V such that

(1) $\{v\} \in X \quad \forall\, v \in V$

(2) If $\Delta \in X$ and $\Delta' \subset \Delta$, then $\Delta' \in X$.

An element $\Delta \in X$ is called an (abstract) simplex. If $\Delta' \subsetneq \Delta$, then Δ' is a face of Δ. Define $\dim(\Delta) = |\Delta| - 1$, and Δ is a k-simplex if $\dim(\Delta) = k$.

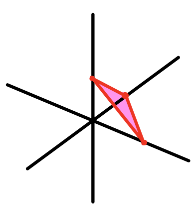
A 0-simplex is a single vertex $\{v\}$. A 1-simplex we call an edge (a pair $\{v, w\}$).

The k-skeleton is $X^{(k)} = \bigcup_{\substack{\Delta \in X \\ \dim(\Delta) \le k}} \Delta$, and $\dim(X) = \max\{\dim(\Delta) : \Delta \in X\}$

If $\dim(X) < \infty$, then we say X is finite dimensional.

The standard n-simplex Δ^n is the convex hull of the standard basis $e_1, \dots, e_{n+1}$ in $\mathbb{R}^{n+1}$.

E.g. in $\mathbb{R}^3$:



To an abstract simplicial complex, we can associate a "simplicial cell complex"

n-simplex $\Delta \longmapsto$ standard n simplex

Δ' a face of $\Delta \longmapsto$ glue accordingly.

$V = V(X) = X^0$: vertex set of X,

$\Delta \subseteq V$ an abstract simplex if $\longleftarrow$ X

Δ span a standard simplex

Aim of this section: define the basic construction $\mathcal{U}$ of (W,S) a Coxeter system.

6.2. Definition: if (W,S) a Cox. system, and X a connected, Hausdorff topological space, a mirror structure on X over S is a family $(X_s)_{s\in S}$ of closed, nonempty subsets of X. X is called a mirrored space over S, and X_s is the s-mirror of X.

6.3 Remark: There is a more general definition for G any group and S indexing families of subgroups (see Davis 5.1)

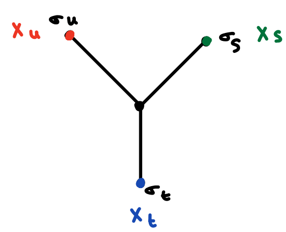
$\mathcal{U}(W,X)$ is obtained by gluing $|W|$ copies of X along mirrors

6.4. Definition: if (W,S) Cox. system and X a mirrored space over S, then the nerve of X is denoted $N(X)$ and is an abstract simplicial complex with vertex set S and $T \subseteq S$ is a simplex iff $\bigcap_{t\in T} X_t \neq \emptyset$

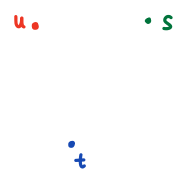
6.5 Examples:

(1) $X = \text{Cone}\,\{\sigma_s \mid s \in S\}$ i.e. star graph with valence $|S|$
$X_s = \{\sigma_s\}$.

e.g. if $S = \{s, t, u\}$, then



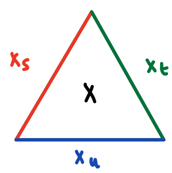
and Nerve:



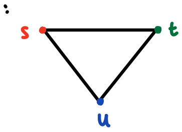
(2) $X = \Delta^n$, with $|S| = n+1$. Then we have $|S|$ codimension one faces, labelled by S

$\{\Delta_s : s \in S\}$, $X_s = \Delta_s$

let $S = \{s, t, u\}$.



Nerve:



(3) P^n convex polytope in $\mathbb{X}^n$. when $n \geq 2$, then $\{F_i\}_{i\in I}$ faces, if $i \neq j$ then $F_i \cap F_j = \emptyset$ $(m_{ij} = \infty)$ or meet at an angle between them π/m_{ij}, $m_{ij} \geq 2$, and set $m_{ii} = 1$. Then (W,S) (isometry group) is the Coxeter system with matrix $[m_{ij}]$. Then take $X = P^n$, and $X_{s_i} = F_i$.
$X_{s_i} = X_i$?

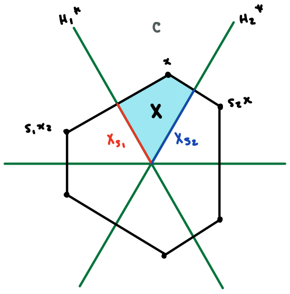
mirrored spaces are the half planes.

03/03.

4) $C \subseteq V^*$ the chamber (closed intersection of half spaces from hyperplanes) associated to Tits representation. We take H_i^* dual hyperplane fixed by $\sigma_i := \rho^*(s_i)$. Then take $X = C$, $X_{s_i} = C \cap H_i^*$.

5) If W is finite, $V^* \rightsquigarrow \mathbb{E}^n$, and $C = \{ v \in \mathbb{R}^n \mid \langle v, e_i \rangle \geq 0 \ \forall i \}$. Then $x \in C^\circ$, Coxeter polytope is Wx, the orbit. Then take $X = C \cap$ Coxeter polytope, and $X_{s_i} = X \cap H_i$ (H hyperplane)

E.g. D_6.



in new not.

remember we defined $\varphi_i := B(e_i, -) = \langle e_i, - \rangle$

and $C_i = \{ \varphi \in V^* : \varphi(e_i) \geq 0 \}$

$= \{ v \in V : \langle v, - \rangle (e_i) \geq 0 \}$ by $V^* \rightsquigarrow \mathbb{E}^n$

$= \{ v \in V : \langle v, e_i \rangle \geq 0 \}$

so $C = \bigcap_i C_i = \{ v \in V : \langle v, e_i \rangle \geq 0 \ \forall i \}$

For the rest of this section, (W, S) is a Cox. system, X mirrored space over S, and $\exists x \in X$ s.t. $x \notin \bigcup_{s \in S} X_s$

Then define $\forall x \in X$ a subset $S(x) = \{ s \in S : x \in X_s \}$ (don't confuse with $S(w)$ from section 5)

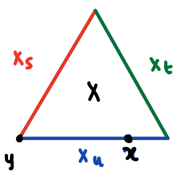
6.6. Examples :

In 6.5 (1), then $S(x) = \begin{cases} \emptyset & x \notin \{ \sigma_s : s \in S \} \\ \{s\} & x = \sigma_s \end{cases}$

In 6.5 (2), then $S(x) = \begin{cases} \emptyset & \text{if } x \in \mathring{X} \\ T \notin S & x \in \bigcap_{t \in T} D_T \end{cases}$

can be just one or two elts. depending on where on simplex X_s it lies

e.g.



$S(y) = \{ s, u \}$

$S(x) = \{ u \}$

6.7. Definition Consider W as a topological space with discrete topology, and $W \times X$ with product topology. Then the basic construction is the topological space with quotient topology

$$\mathcal{U}(W, X) = W \times X / \sim ,$$

where $(w, x) \sim (w', x') \Leftrightarrow x = x'$ and $w^{-1} w' \in W_{S(x)}$. ← parabolic subgroup of W, $W_{S(x)} := \langle S(x) \rangle$

Write $[w, x]$ for equivalence class of (w, x) in $\mathcal{U}(W, X)$

If $x \in X_s$, then $s \in S(x)$, so $(w, x) \sim (ws, x)$ since $w^{-1} w s = s \in W_{S(x)}$. Hence $[w, x]$ contains at least (w, x) and (ws, x).

6.8. Definition : write wX for $\{w\} \times X$ in $\mathcal{U}(W, X)$, for any $w \in W$. Then wX is called a chamber of $\mathcal{U}(W, X)$. The fundamental chamber is eX, which we identify with X. Hence wX and wsX are glued / identified along X_s.

$wX = \{w\} \times X$ under $\sim$, and $wsX = \{ws\} \times X$ under $\sim$. And so by our previous statement, $(w, x) \in wX$ and $(ws, x) \in wsX$, and for $x \in X_s$, $(w, x) \sim (ws, x)$ since $w^{-1} ws = s \in W_{S(x)}$, so wX and wsX are identified along X_s

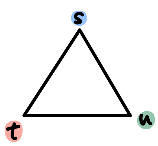
6.9. Lemma: The Cayley graph:

For X as in 6.5 (1), up to subdivision $\mathcal{U}(W,X)$ is $\mathrm{Cay}_S(W)$.

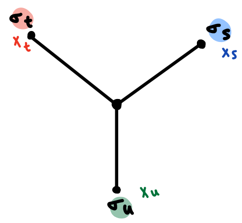
proof: let $x \in X$. Then if $x \notin \{\sigma_s \mid s \in S\}$, we have $W_{S(x)} = W_\emptyset = \{e\}$, so $(w,x) \sim (w',x')$ iff $w^{-1}w' \in \{e\}$ iff $w = w'$. So $[w,x] = \{(w,x)\}$. Otherwise $x \in \{\sigma_s : s \in S\}$, say $x = \sigma_s$ for some $s \in S$. Then $W_{S(x)} = W_{\{s\}} = \{e, s\}$ since s is an involution. Hence $(w,x) \sim (w',x')$ iff $w^{-1}w' \in \{e,s\}$ iff $w = w'$ or $w' = ws$. I.e. $[w,x] = \{(w,x),(ws,x)\}$.

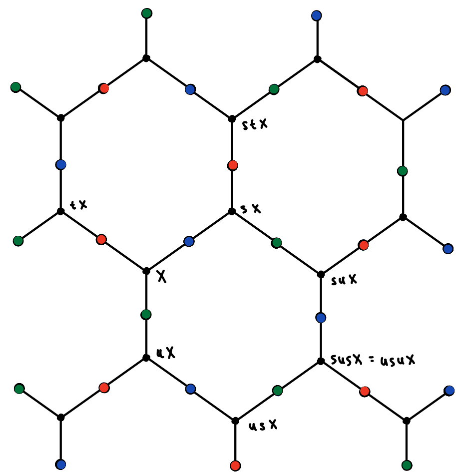
Therefore in $\mathcal{U}(W,X)$, we glue wX and wsX along $X_s = \{\sigma_s\}$, and these are <u>all</u> the gluings. If we label the star points of wX by w, then this gives the Cayley graph $\mathrm{Cay}_S(W)$, and the edges are subdivided by <u>the mirrors</u> σ_s, and mirror labels $\leftrightarrow$ edge labels in $\mathrm{Cay}_S(W)$.

I wrote this.?

E.g. $\Gamma =$ 

, the (3,3,3) triangle group, and X as in 6.5 (1), then.

I.e. $X =$ 

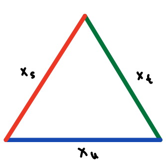


6.10. Definition: For X the mirrored space in example 6.5 (2) i.e. X a simplex with codimension 1 faces $\{D_s \mid s \in S\}$. Then $\mathcal{U}(W,X)$ is called the Coxeter complex.

6.12 Example: Coxeter complex for (3,3,3)-triangle group.

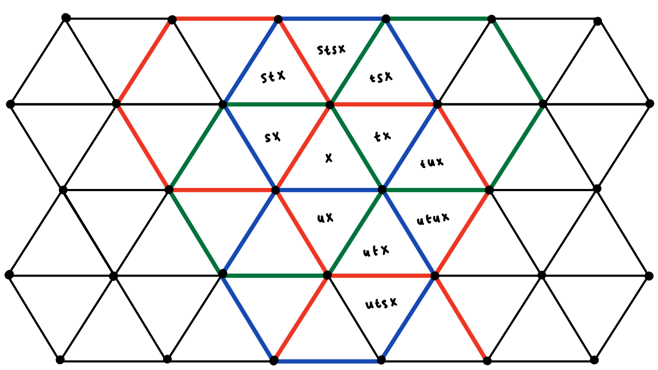
If $x \in X_s \cap X_t$, then $W_{S(x)} = \langle s,t \rangle \cong D_6$. So $\forall\, w \in W$, wX is glued to wX (trivially), wsX, wtX, $wstX$, $wtsX$ and $wstsX$ at $x \in X_s \cap X_t$.

Get picture: $\mathcal{U}(W,X)$ a tesselation of $\mathbb{E}^2$ by triangles:



glue e.g. X to uX along X_u

. . .



. . .

6.11 Remark: If W is an irreducible finite Coxeter group, then Coxeter complex can be identified with the tesselation of the sphere by spherical simplices induced by W.

nonexaminable:
If W is affine, then $\exists$ an affine subspace $E \subset V^*$ given by slicing across the interior of the Tits cone. Then W acts on E by isometries, and Coxeter complex $\longleftrightarrow$ tesselation of $\mathbb{E}^n$ given by intersecting E with the interior of the tits cone.

6.13. Lemma: $\mathcal{U}(W,X)$ is a connected topological space

pf: $\mathcal{U}(W,X)$ has a quotient topology $\Rightarrow$ wts only subsets that are both open and closed are $\emptyset$ and $\mathcal{U}(W,X)$.
Suppose $A \subseteq \mathcal{U}(W,X)$ is open (closed resp.). Then by dfn of quotient topology, A is open iff $A \cap wX$ is open $\forall w \in W$.

Let $A \neq \emptyset$, and assume that A is both open and closed. Since X is connected, $\Rightarrow A \cap wX = wX$ or $= \emptyset$.
To see this, consider that A^c, the complement of A, is also open and closed. If $A \subset wX$ (strict), so $A^c \cap wX \neq \emptyset$, then this says wX can be written as a disjoint union of nonempty open sets, $A \cup A^c$, which violates the connectedness.

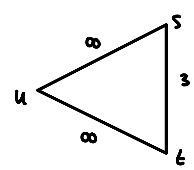
So then considering this $\forall w \in W$, $\Rightarrow$ A is a union of chambers, i.e. $A = \bigcup_{v \in V} vX$, $\emptyset \neq V \subseteq W$ (under our assumption $A \neq \emptyset$). For $v \in V$, $s \in S$, $\exists x \in X_s \neq \emptyset$ (from way back), so that $[vs, x] = [v, x]$ ($v^{-1}vs = s \in W_{S(x)}$).

Hence $\exists$ an open neighbourhood in A at x intersecting vsX. Hence, $A \cap vsX \neq \emptyset$, and so $vs \in V \Rightarrow VS \subseteq V$.
But $\langle S \rangle = W$, $\Rightarrow V = W$, $\Rightarrow A = \mathcal{U}(W,X)$. So the only open and closed subsets of $\mathcal{U}(W,X)$ are $\emptyset$ and $\mathcal{U}(W,X)$ itself. $\square$

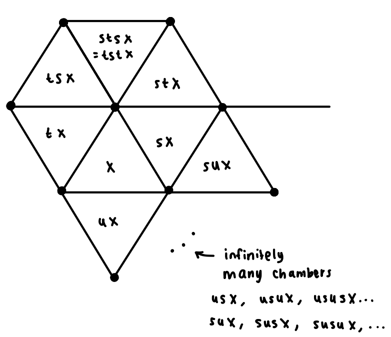
6.14. Definition $\mathcal{U}(W,X)$ is said to be locally finite if $\forall [w,x] \in \mathcal{U}(W,X)$, there is an open nhood which meets only finitely many chambers.

6.15. Examples

- $Cay_W(S)$ (Ex. 6.9) as $\mathcal{U}(W,X)$ is locally finite
- The Coxeter complex (definition 6.11) is not necessarily locally finite.
  Example 6.12 is, but $(W(\Gamma), S)$ given by Γ below is not: $\Gamma =$



To see this, consider that if $x \in X_s \cap X_u$, then $W_{S(x)} \cong D_\infty$, so $\exists$ infinitely many chambers wX, $w \in \langle s,u \rangle$, glued to eX at (e,x).



$W_{S(x)} \cong D_\infty$, looks like ususus$\cdots$
sususu$\cdots$

So we glue usX to $usuX$ to $ususX$ ...
along uX sX

and $uX \cap sX = x$ say, then x intersects all of these (infinitely many) chambers.

6.16. Lemma: The following are equivalent:

(1) $\mathcal{U}(W,X)$ is locally finite

(2) $\forall x \in X$, $W_{S(x)}$ is finite

(3) $\forall T \subseteq S$ such that W_T is infinite, then $\bigcap_{t \in T} X_t = \emptyset$.

Pf: Clearly (2) $\Leftrightarrow$ (3): Remember that $S(x) = \{ s \in S : x \in X_s \}$. So if (2) holds, then if $\exists x \in \bigcap X_t$, then $T \subseteq S(x)$, and $W_T \subseteq W_{S(x)}$ $\Rightarrow$ $W_{S(x)}$ infinite. Then if (3) holds, if $x \in X$, then any set of mirrors $\{X_s\}$ $T \subseteq S$ containing X must have W_T finite. In particular, the set $S(x)$ of all mirrors containing x will have $W_{S(x)}$ finite.

(1) $\Rightarrow$ (3): Suppose that (3) does not hold. Then $\exists x \in X$ with $W_{S(x)}$ infinite. So in $\mathcal{U}(W,X)$, an infinite number of chambers are identified at $[e,x]$, so $\mathcal{U}(W,X)$ by dfn is not locally finite. So (1) does not hold if (3) does not hold.

> If 3 does not hold, then $\exists T \subseteq S$ s.t W_T infinite (unless all finite in which case (1) automatically true) with $\bigcap_{t \in T} X_t \neq \emptyset$. So $\exists x \in \bigcap_{t\in T} X_t$. But $S(x) = \{ s \in S : x \in X_s \}$, so certainly we can bulk up T (if necessary) to include all of $S(x)$ (by dfn of $S(x)$, $T \subseteq S(x)$). Then $T \subseteq S(x)$ and W_T infinite $\Rightarrow$ $W_{S(x)}$ infinite.

(2) $\Rightarrow$ (1): For each $[w,x] \in \mathcal{U}(W,X)$, $\exists$ an open neighbourhood $\mathcal{U}$ of $[w,x]$ only intersecting chambers $w'X$ where $w^{-1}w' \in W_{S(x)}$ (generally true). But if (2) holds, then $|W_{S(x)}| < \infty$, so $\mathcal{U}$ only intersects a finite number of chambers. So $\mathcal{U}(W,X)$ is locally finite and (1) holds. $\square$

Remark that W acts on $\mathcal{U}(W,X)$ by homeomorphisms via a left action on W, X:

$$\text{i.e.} \qquad w' \cdot (w,x) = (w' \cdot w, x)$$

This clearly preserves the equivalence relation, so we get an action on $\mathcal{U}(W,X)$.

> If $[w,x] = [v,x]$, then $w^{-1}v \in W_{S(x)}$. So applying w', note that $(w'w)^{-1}(w'v) = w^{-1}(w')^{-1}w'v = w^{-1}v \in W_{S(x)}$, so that $[w'w,x] = [w'v,x]$

(Recall Definition 1.7 on strict fundamental domain for $G \curvearrowright X$).

6.17 Lemma: The fundamental chamber is a strict fundamental domain for $W \curvearrowright \mathcal{U}(W,X)$ $\Rightarrow$ $\mathcal{U}(W,X)/W \cong X$

Moreover, we have $w' \cdot (wX) = w'wX$, which gives a transitive, free action of W on the set of chambers of $\mathcal{U}(W,X)$. So

only element sending wX to wX is e, since W has a free action on a point $x \notin \bigcup X_s$ (by ass. x exists)

$$\left.\begin{array}{ccc} W & \longrightarrow & \mathcal{U}(W,X) \\ w & \longmapsto & wX \end{array}\right\} \text{is a bijection}$$

6.18 Lemma: $\mathrm{Stab}_W([w,x]) = \{ w' \in W : w^{-1}w'w \in W_{S(x)} \}$ just by dfn of stab $= wW_{S(x)}w^{-1}$

> By dfn, $\mathrm{stab}_W([w,x]) = \{ v \in W : v \cdot [w,x] = [w,x] \}$
> $= \{ v \in W : [vw,x] = [w,x] \}$ by dfn of action
> $= \{ v \in W : w^{-1}vw \in W_{S(x)} \}$ by dfn of equivalence relation

WTS $= wW_{s(x)}w^{-1}$. First show $wW_{s(x)}w^{-1} \subseteq$ stab: Say $u \in W_{s(x)}$. Then $x \in X_u$, and so consider $v := wuw^{-1}$.
Then $[vw, x] = [wuw^{-1}w, x] = [wu, x]$, and $\forall u \in s(x)$, $[w, u] = [wu, x]$. so $wW_{s(x)}w^{-1} \subseteq$ stab.

Also by dfn, $\forall v \in$ stab, $w^{-1}vw \in W_{s(x)}$, $\Rightarrow$ w^{-1} stab $w \subseteq W_{s(x)} \Leftrightarrow$ stab $\subseteq wW_{s(x)}w^{-1}$.

6.19 Lemma: The space $\mathcal{U}(W,X)$ is Hausdorff.

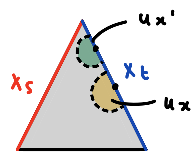
proof: let $y = [w,x] \in \mathcal{U}(W,X)$, $W_y = \mathrm{stab}_W([w,x])$. Then for $x \in U_x \subset X$ an open neighbourhood,

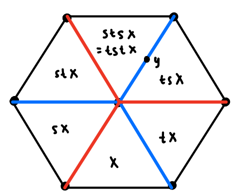
$$V_y := W_y \left(U_x \setminus \bigcup_{x \notin X_s} X_s\right)$$

mirrors are closed, U_x is open and action of W takes open to open

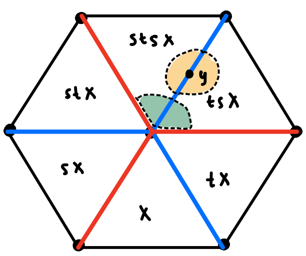
is open in $\mathcal{U}(W,X)$. If $y' = [w', x']$ is such that $y \neq y'$, then we can choose U_x and $U_{x'}$ small enough to have $V_y \cap V_{y'} \neq \emptyset$. $\square$

E.g. $\Gamma = s \overset{3}{—} t$, then $X =$





If $y = [ts, x]$, then $W_y = ts\, W_t\, (ts)^{-1} = ts\, W_t\, st$
$y' = [ts, x']$ $= \{e, s\}$ by direct calculation using $s(y) = t$ (lies in X_t)



6.20 Definition: if G is a discrete group, and Y is a Hausdorff space, then an action by homeomorphisms $G \curvearrowright Y$ is properly discontinuous if
(i) Y/G is Hausdorff
(ii) $\forall y \in Y$, $G_y = \mathrm{Stab}_G(y)$ is finite
(iii) $\forall y \in Y$, $\exists$ an open nhood U_y of Y s.t $G_y \cdot U_y = U_y$ (stabilizes open nhood of y, but not necessarily pointwise) and $gU_y \cap U_y = \emptyset$ $\forall g \notin G_y$.

6.21 Lemma: The W action on $\mathcal{U}(W,X)$ is properly discontinuous iff $W_{s(x)}$ are spherical (finite) $\forall x \in X$.

proof: ($\Leftarrow$) (i) and (ii) are immediate by 6.18, 6.19. ($\mathcal{U}(W,X)/W \simeq X$) For (iii) wlog we'll show it for $[e,x]$. Then

$$V_y := W_{s(x)}\left(X \setminus \bigcup_{x \notin X_s} X_s\right)$$

$\Leftarrow$ (i) and (ii): $\mathcal{U}(W,X)/W \simeq X$ and X by ass. is Hausdorff and $\mathrm{Stab}_W([w,x]) = wW_{s(x)}w^{-1}$

from 6.19 satisfies $W_y V_y = V_y$, and $wV_y \cap V_y = \emptyset$ $\forall w \in W \setminus W_{s(x)}$.

($\Rightarrow$) part (ii) of definition 6.20 says that $\mathrm{stab}_W([w,x]) = wW_{s(x)}w^{-1}$ (lemma 6.18) is finite. But if this is finite, then so is $W_{s(x)}$ (they're conjugate and so have the same cardinality.

7. The Davis Complex

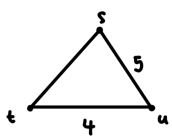
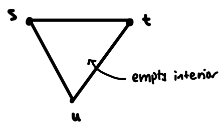
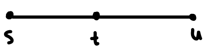
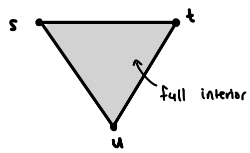
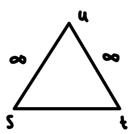
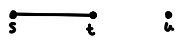
Recall $\mathcal{S} = \{ T \subseteq S : W_T \text{ is spherical (finite)} \} \ni \emptyset, \quad W_\emptyset = \{e\}.$

7.0 Remark: In this section, abstract simplicial complexes do not have $\emptyset$ simplex.

7.1 Definition: The nerve of (W,S) denoted $L(W,S)$, is an abstract simplicial complex with vertex set S and simplex set $\mathcal{S} \setminus \{\emptyset\}$

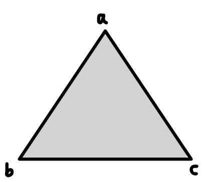
if W_T spherical, then $\emptyset \neq P \subseteq T \Rightarrow W_P$ spherical so $P \subseteq \mathcal{S} \setminus \{\emptyset\}$

7.2 Examples

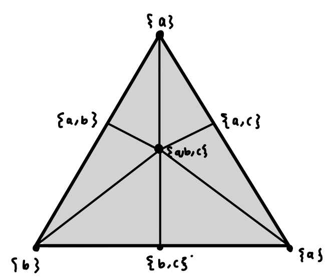
	Γ	$L(W(\Gamma), S(\Gamma))$	
1)			$\mathcal{S} = \{\{s\}, \{t\}\}$, and $\{s,t\} \notin \mathcal{S}$ b.c. $W_{\{s,t\}}$ has infinite order ($\cong D_\infty$)
2)			the graph is not a finite type graph, and so must generate $(\{s,t,u\})$ an infinite Coxeter group
2.5)			remember no edge between s_i and s_j says that $(s_i s_j)^2 = e$. This is A_3, which is one of our finite coxeter groups.
3)	 $W \cong D_\infty \times C_2$		
4)	$S = S_1 \sqcup S_2$ w/ $m_{ij} = \infty$ $\forall$ $s_i \in S_1$, $s_j \in S_2$, then let $W_i := \langle S_i \rangle$, $i = 1,2$.	$L(W,S) = L(W_1, S_1) \sqcup L(W_2, S_2)$	

7.3. Definition: Given an abstract simplicial complex X, it's barycentric subdivision is the a.s.c. X' with vertex set X, and simplex set $X' = \{ \{\Delta_0, \dots, \Delta_p\} : \Delta_i^{p \in \mathbb{N}} \subset \Delta_{i+1} \ \forall\, 0 \leq \dots \leq p-1 \}$

e.g. $X = \{a,b,c\}$



Then.



see for example $\{a\} \subseteq \{a,c\} \subseteq \{a,b,c\}$

each included guy is a face of the next, and we do all possibilities.

simplex set of X: $\left. \begin{array}{l} \{a\}, \{b\}, \{c\}, \\ \{a,b\}, \{b,c\}, \{a,c\} \\ \{a,b,c\} \end{array} \right\}$

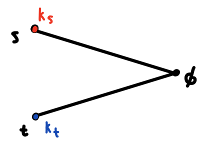
7.4. Definition: The chamber K of (W,S) is the cone on the barycentric subdivision L' of the nerve $L = L(W,S)$.
Let $K_s \subset K$ be the star in L' of the vertex s, $K_s = \bigcup_{\substack{\sigma_p \in L' \\ s \in \sigma_p}} \sigma_p$. We label the cone point "$\emptyset$".

union over all simplices σ_p in the Barycentric subdivision of the nerve $L(W,S)$, L', where s is a vertex in σ_p

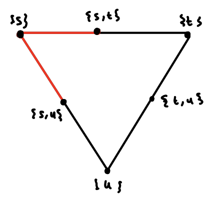
7.5 Examples (from 7.2)

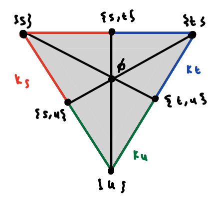
$L'(W,S) = L'$ $\qquad$ K

1)



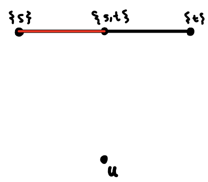
2)

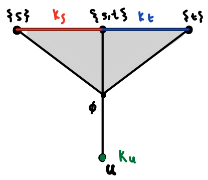




taking the cone fills it in to a 2-simplex

3)





7.6. Remarks:

- K is connected and Hausdorff, so $\{K_s\}$ is a mirror structure on K.
- K is the a.s.c. $\text{Flag}(\mathcal{S})$ (think of $\mathcal{S}$ as a poset with inclusion)
- $\emptyset$ is contained in no mirror $\quad$ we need some point $x \notin \bigcup_{s\in S} X_s$ for the basic construction to not be degenerate
- In all our examples, K_s is 1-dimensional, but this is not always the case.

7.8. Lemma: a mirrored space X for (W,S) satisfies $W_{s(x)}$ finite $\forall x \in X \iff N(X) \le L(W,S)$

proof: ($\Rightarrow$) Let $\{t_1,\dots,t_k\}$ be a simplex in $N(X)$. Then $\exists\ x \in \bigcap_{i=1}^{k} X_{t_i}$, so $W_{\{t_1,\dots,t_k\}} \subseteq W_{s(x)}$ is finite,
so $\{t_1,\dots,t_k\} \in \mathcal{S}$, $\Rightarrow$ $\{t_1,\dots,t_k\}$ is a simplex in $L(W,S)$.
So $N(X) \subseteq L(W,S)$

(ass)

($\Leftarrow$) Let $x \in \bigcap_{t\in T} X_t \Rightarrow T$ is a simplex in $N(X) \Rightarrow T$ is a simplex in $L(W,S) \Rightarrow T \in \mathcal{S} \Rightarrow W_T = W_{s(x)}$ is finite.

$\square$

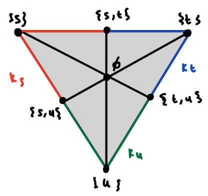
For the reverse implication, really we want to say that $N(X) \le L(W,S) \Rightarrow W_{s(x)}$ is finite. Now if $x \notin \bigcup_s X_s$, then we know that $s(x) = \emptyset$, and $W_\emptyset = e$ so $W_{s(x)}$ is finite. Now suppose that $x \in X_s$ for some s. Then in fact $x \in \bigcap_{s \in s(x)} X_s$, and so $s(x)$ is a simplex in $N(X) \Rightarrow s(x)$ is a simplex in $L(W,S)$ (by assumption) $\Rightarrow s(x) \in \mathcal{S}$, $\Rightarrow W_{s(x)}$ is finite.

$\square$

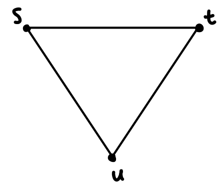
recall $\Big\{$ **6.4. Definition:** if (W,S) cox. system and X a mirrored space over S, then the nerve of X is denoted $N(X)$ and is an abstract simplicial complex with vertex set S and $T \subseteq S$ is a simplex iff $\bigcap_{t\in T} X_t \neq \emptyset$ $\Big\}$

7.9 Corollary: K satisfies $N(K) = L(W,S)$, so $W_{S(x)}$ is finite $\forall\, x \in K$ (ES 3).

General idea: you take (W,S), and define the Nerve of (W,S), to be the abs. simp. compl. $L(W,S)$ with vertex set S and simplex set $\{T \subseteq S : W_T \text{ spherical}\}$. We can do barycentric subdivision on $L(W,S)$ to get $L'(W,S)$, and then take the cone on L'. Label cone point $\emptyset$. This space we call the chamber of (W,S), and denote it by K. We can define a mirror structure on K by looking at the (simplicial) star on each vertex s, which is closed and gives us s-mirror K_s. Now we want to think about the nerve of K. E.g. take K to be:



Then our vertex set is $\{s,t,u\}$, and we see that $K_s \cap K_t$, $K_s \cap K_u$, $K_u \cap K_t$ are all nonempty and $K_s \cap K_t \cap K_u$ is empty. So we recover our original nerve:



This is essentially just reversing the construction. The second part follows as a corollary of the lemma.

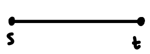
7.10 Definition: The Davis complex $\Sigma(W,S) = \mathcal{U}(W,K)$.

7.11 Corollary: $\Sigma(W,S)$ is connected, Hausdorff, locally finite, and W acts properly discontinuously on Σ with quotient K. All point stabilizers are conjugates of spherical subgroups of W.

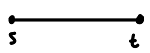
Follows from lemmas 6.18, 6.19, and 6.21. + fact that $W_{S(x)}$ is finite $\forall\, x \in K$.
i.e. so all point stabilizers are finite

7.12 Examples:

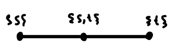
D_6



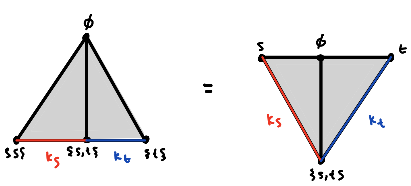
L



L'

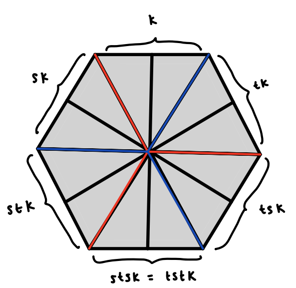


K



(redrawing)

$\Sigma(W,S)$

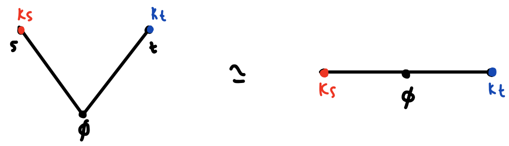


D_∞:

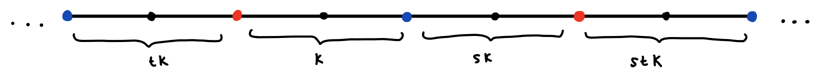
L

L'

K



$\Sigma(W,S) = \mathcal{U}(W,K)$



3) W is the $(3,3,3)$ triangle group, then $\Sigma(W,S)$ is the barycentric subdivision of the tiling of $\mathbb{E}^2$ by triangles

Remark 7.13: If W is a Euclidean or hyperbolic geometric reflection group, then $\Sigma(W,S)$ is the barycentric subdivision of the corresponding tesselation of $\mathbb{E}^n$ or $\mathbb{H}^n$ by P.

If W is finite (spherical) then $\Sigma(W,S)$ can be identified with the Barycentric subdivision of the associated Coxeter polytope

The remainder of the course is devoted to proving the following theorem:

Thm 7.14: The Davis Complex $\Sigma = \Sigma(W,S)$ is contractible.

Definition 7.15: For $w \in W$, define $\mathrm{In}(w) = \{ s \in S \mid \ell_S(ws) < \ell_S(w)\}$
$\mathrm{Out}(w) = \{ s \in S \mid \ell_S(ws) > \ell_S(w)\}$

Example: s —— t

$w = sts$. Then
$\mathrm{In}(w) = \{s,t\}$
$\mathrm{Out}(w) = \emptyset$

b.c. $ws = stss = st$
$wt = stst = tstt = ts$.
(length shortened)

Remark 7.16:

- $\ell_S(ws) = \ell_S(w) \pm 1$, so $S = \mathrm{In}(w) \sqcup \mathrm{Out}(w)$.
- If $\ell_S(ws) < \ell_S(w)$, then if $(s_1,\dots,s_k)$ a reduced word for w, we have that by (E) on $(ws)^{-1}$ that $w^{-1} = s\, s_k \cdots \hat{s}_i \cdots s_1 \Rightarrow w = s_1 \cdots \hat{s}_i \cdots s_k s$.

Note that if $(s_1,\dots,s_k)$ a reduced word for w, then $(s_k,\dots,s_1)$ a reduced word for w^{-1}. So $\ell_S(w) = \ell_S(w^{-1})$. In particular $\ell_S(ws) = \ell_S((ws)^{-1}) = \ell_S(s^{-1}w^{-1}) = \ell_S(sw^{-1})$. Now If $\ell_S(ws) < \ell_S(w)$, then $\ell_S(sw^{-1}) < \ell_S(w^{-1})$, So by (E) $\exists$ i s.t $w^{-1} = s\, s_k \cdots \hat{s}_i \cdots s_1 \Leftrightarrow w = s_1 \cdots \hat{s}_i \cdots s_k s$.

So actually, $\mathrm{In}(w) = \{ s \in S \mid \exists$ a reduced word for w ending in $s\}$ (∗)

The above says that $\mathrm{In}(w) \subseteq (*)$. But clearly if $\exists$ reduced word for W ending in s, then $\ell_S(ws) < \ell_S(w)$, so $(*) \subseteq \mathrm{In}(w) \Rightarrow \mathrm{In}(w) = (*)$.

finite word — left multiplying

Lemma 7.17: (W,S) Coxeter system. Suppose $\exists\, w_0 \in W$ such that $\ell_S(sw_0) < \ell_S(w_0)$ $\forall s \in S$. Then W is finite.

pf: Suppose W is infinite, and let $\underline{s} = (s_1, \dots, s_i, \dots)$ be a possibly infinite sequence of elements in S. Let $\underline{s_i} = (s_i, \dots, s_1)$ be an initial subsequence in reverse, and assume each $\underline{s_i}$ is reduced. We assume for contradiction's sake that $\exists\, w_0 \in W$ s.t. $\ell_S(sw_0) < \ell_S(w_0)$ $\forall s \in S$.

Claim: w_0 has a reduced expression starting with $\underline{s_i}$ $\forall i$.

Assuming the claim, we get that $\forall\, u \in W$, $\ell(w_0) = \ell(u) + \ell(u^{-1}w_0)$

because we took this arbitrary sequence which is possibly infinite, and so any reduced word for u in W appears as a starting section of a sequence of the form $\underline{s}$. So if $(s_1, \dots, s_i)$ say is a word for u, then this is reduced and so if $w_0 = (\underline{s_i}, \underline{y})$ is reduced, then $u^{-1}w_0$ has reduced word a, and so $\ell(w_0) = \ell(\underline{s_i}) + \ell(a) = \ell(u) + \ell(u^{-1}w_0)$.

Since $\ell_S(u^{-1}w_0) \geq 0$, $\Rightarrow$ $\ell(w_0) \geq \ell(u)$, so $\ell(u)$ is bounded $\forall\, u \in W \Rightarrow W$ finite (remember W is finitely generated). This is a contradiction (we assumed W infinite), and so there can be no such $w_0 \in W$. We've thus (almost) proved the contrapositive. We just need to show the claim is true.

proof of claim: base case: when $i = 1$, true by the exchange condition. We know that $\forall s \in S$, $\ell_S(sw_0) < \ell_S(w_0)$, and so we can exchange some t_i in a reduced word for w_0 for s placed at the beginning: i.e. if $(t_1, \dots, t_k)$ a reduced word for w_0, then $w_0 = s t_1 \cdots \hat{t_i} \cdots t_k$, and $(s, t_1, \dots, \hat{t_i}, \dots, t_k)$ a reduced word for w_0 still.

Inductive hypothesis: assume that w_0 has a reduced expression starting with $\underline{s_{i-1}}$. Then WTS that has one starting with $\underline{s_i}$. Now again by the exchange condition, $\ell_S(s_i w_0) < \ell_S(w_0)$, so we have that we can exchange some t in the reduced expression starting with $\underline{s_{i-1}}$ for s_i placed at the beginning. The idea is then that if we have say w_0 has reduced word $(s_{i-1}, s_{i-2}, \dots, s_1, y_1, \dots, y_k)$, then we cannot omit one of the $\hat{s_j}$'s. Say t is in the initial string, then

$$s_{i-1} \cdots s_1 (\_\_) = s_i \cdots \hat{s_j} \cdots s_1 (\_\_)$$

Then cancelling the stuff in the brackets and up to s_{j-1} gives us

$$s_{i-1} \cdots s_j = s_i \cdots s_{j+1}$$

But this means that $\underline{s_i} = (s_i, \dots, s_1) = (s_{i-1}, \dots, s_j, s_j, \dots, s_1)$, which is not reduced. ↯. So we exchange s_i for a t after $\underline{s_{i-1}}$, and the claim holds. $\square$

Lemma 7.18: For $T \subset S$, there is a unique element w of minimal length in the coset wW_T, such that all elements $w' \in wW_T$ can be written in the form $w' = wa$, $a \in W_T$ such that

$$\ell_S(w') = \ell_S(w) + \ell_S(a)$$

proof: Let w be a minimal length element in wW_T. This looks a little funny but is okay, just think about the fact that wW_T is a coset, and so can be written in multiple different ways. Anyways but if $u \in wW_T$, then $u = wg$ for some $g \in W_T$, so $uW_T \subseteq wW_T$. Also $w = ug^{-1}$, and so $wW_T \subseteq uW_T \Rightarrow wW_T = uW_T$. Alright so we don't have to stress too much about the notation.

Suppose w has min. length in wW_T. Write w' in wW_T as wb with $b \in W_T$. Let $\underline{u}$ and $\underline{s}$ be reduced words for w and b respectively. Then $\underline{u}\,\underline{s}$ (concatenation), is a word for w' - perhaps not reduced. Suppose $\underline{u}\,\underline{s}$ is not reduced.

Then (D) $\Rightarrow$ can delete two letters in $\underline{u}$ and $\underline{s}$. Both cannot be in $\underline{u}$, since w had minimal length. Both cannot be in $\underline{s}$, or $\underline{s}$ not reduced. Also cannot have one in $\underline{u}$ and one in $\underline{s}$, as then multiplying by $(s_1 \dots \hat{s_i} \dots s_k)^{-1}$ on RHS gives you a shorter word than w in the coset ($\underline{u}$ with one guy removed).

$\Rightarrow$ $\underline{u}\,\underline{s}$ is reduced, and hence $\ell(w') = \ell(w) + \ell(b)$.

To see uniqueness, now suppose two such elements v and w are in wW_T, $\ell(v) = \ell(w)$. Then $v = wb$ for some $b \in W_T$, and $\ell(wb) = \ell(w) + \ell(b) = \ell(v)$. $\Rightarrow \ell(b) = 0$ so $b = e$ $\Rightarrow v = w$.

$\square$

Proposition 7.19: $\forall w \in W$, $In(w) \in \mathcal{S}$, i.e. $W_{In(w)}$ is finite.

proof: consider the coset $wW_{In(w)}$, and let u be the unique element of minimal length. Then by lemma 7.18, $w \in wW_{In(w)}$ can be written as $w = ua$, $a \in W_{In(w)}$, s.t

$$\ell_S(w) = \ell_S(u) + \ell_S(a) \quad (*)$$

Now $\forall s \in In(w)$, we have $\ell_S(ws) < \ell_S(w)$ (†) by dfn of $In(w)$. Also for $s \in In(w)$, $as \in W_{In(w)}$, so we have $ws = uas$ satisfies

$$\ell_S(ws) = \ell_S(u) + \ell_S(as) \quad (**)$$

So from (†) we get

$$\ell_S(u) + \ell_S(as) \overset{(**)}{<} \ell_S(u) + \ell_S(a)$$
$$\Leftrightarrow \quad \ell_S(as) < \ell_S(a) \qquad \forall s \in In(w)$$
$$\Leftrightarrow \quad \ell_S(sa^{-1}) < \ell_S(a^{-1}) \qquad \forall s \in In(w)$$

So we're in the position of lemma 7.17: we have $(W_{In(w)}, In(w))$ a Coxeter system (simply using the fact that $In(w) \subseteq S$), and $\exists\, a^{-1} \in W_{In(w)}$ (since $a \in W_{In(w)}$) s.t $\forall s \in In(w)$,

$$\ell_S(sa^{-1}) < \ell_S(a^{-1}).$$

So lemma 7.17 says that $W_{In(w)}$ is finite, i.e. $In(w) \in \mathcal{S}$. $\square$

Lemma 7.20: The chamber K of (W,S) is contractible and for $T \in \mathcal{S}$, $K^T := \bigcup_{t \in T} K_t$ is also contractible.

pf: remember we said that K was the cone on the barycentric subdivision of the nerve of (W,S), $L(W,S)$. The cone on anything is contractible to the cone point, so K is contractible.

Now for $\emptyset \neq T \in \mathcal{S}$, T spans a simplex σ_T in L (look back at dfn of L, has simplices set $\mathcal{S} \setminus \{\emptyset\}$, and if $T \in \mathcal{S}\setminus\{\emptyset\}$, then the elements of T are the vertices of the simplex σ_T in L). Let σ_T' denote the barycentric subdivision of σ_T in L'. Then σ_T' is contractible (also can think about some more think about it as a cone), so to show that K^T is contractible, its enough to construct a deformation retraction $r: K^T \to \sigma_T'$. Now a vertex $x \in K^T$ means that x lies in some K_t, which corresponds to subsets $T' \in S$ s.t $t \in T'$. In particular, $T' \cap T \neq \emptyset$, so we can map $x \in K^T$ to a vertex of σ_T' corresponding to $T' \cap T$

We extend this to simplices by mapping simplex σ_v with vertices $\{v_0, \dots, v_k\}$ to simplex $\{T \cap v_0, \dots T \cap v_k\}$

$(\sigma_v \in K^T \iff \exists\, T \ni t \in V_i \;\; \forall\, 0 \leq i \leq k)$ need to go over proof. $\square$

Proof of Theorem 7.14:

Recall... **Theorem 7.14:** The Davis complex $\Sigma(W,S)$ is contractible.

List the elements of W as $w_1, w_2, w_3, \dots$ such that $\ell_S(w_n) \leq \ell_S(w_{n+1})\ \forall n \geq 1$. If W is finite, then repeat the last element so we have an infinite list.

Let $U_n = \{w_1, \dots, w_n\} \subseteq W$, so $W = \bigcup_{n=1}^{\infty} U_n$.

Let $P_n = \bigcup_{w \in U_n} wK = \bigcup_{i=1}^{n} w_i K \leq \Sigma(W,S)$.

So $P_i \leq P_{i+1}$ and $\Sigma = \bigcup_{i=1}^{\infty} P_i$.

Now $P_n = P_{n-1} \cup w_n K$

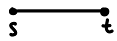
glue along mirrors.

need to think about this

Which mirrors do we glue along? $\{K_s \mid \ell(w_n s) < \ell(w_n)\} = \{K_s \mid s \in In(w_n)\}$. So we glue along $K^{In(w_n)}$.
By proposition 7.19, $In(w) \in \mathcal{S}$, so by lemma 7.20, $K^{In(w_n)}$ is contractible.

We have $P_0 = K$ is contractible, and to get P_n from P_{n-1} we glue on $w_n K$ (which is contractible since K is contractible and $W \curvearrowright K$ preserves structure of K), along $K^{In(w_n)}$, which is contractible.

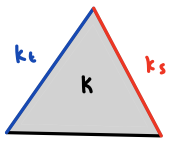
$\Rightarrow$ at each stage P_n is contractible $\Rightarrow$ Σ is contractible. $\square$

Example 7.21: $\Gamma =$ 

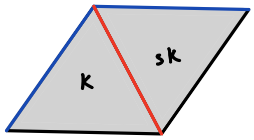
D_6.

List elements: $e, s, t, st, ts, sts = tst$

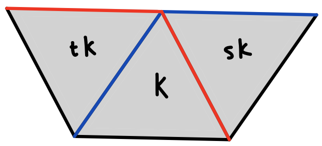
$P_0 = K$ which looks like



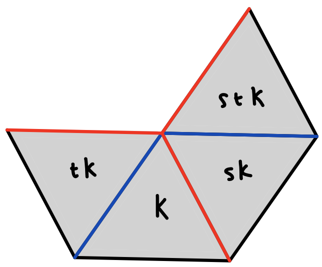
$In(s) = \{s\}$, so glue: get P_1:



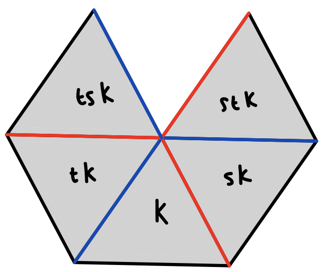
$In(t) = \{t\}$, so glue: get P_2:



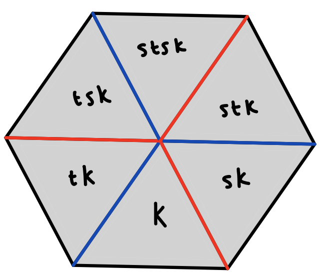
$In(st) = \{t\}$, get P_3:



$In(ts) = \{s\}$, get P_4:



$In(sts) = \{s,t\}$, glue along k_t and k_s, get P_5:



and $P_5 = \Sigma(W,K)$, so done!